

Refined Estimates on Conjectures of Woods and Minkowski

Leetika Kathuria* and Madhu Raka

Centre for Advanced Study in Mathematics

Panjab University, Chandigarh-160014, INDIA

Abstract

Let Λ be a lattice in \mathbb{R}^n reduced in the sense of Korkine and Zolotareff having a basis of the form $(A_1, 0, 0, \dots, 0), (a_{2,1}, A_2, 0, \dots, 0), \dots, (a_{n,1}, a_{n,2}, \dots, a_{n,n-1}, A_n)$ where A_1, A_2, \dots, A_n are all positive. A well known conjecture of Woods in Geometry of Numbers asserts that if $A_1 A_2 \cdots A_n = 1$ and $A_i \leq A_1$ for each i then any closed sphere in \mathbb{R}^n of radius $\sqrt{n}/2$ contains a point of Λ . Woods' Conjecture is known to be true for $n \leq 9$. In this paper we give estimates on the Conjecture of Woods for $10 \leq n \leq 33$, improving the earlier best known results of Hans-Gill et al. These lead to an improvement, for these values of n , to the estimates on the long standing classical conjecture of Minkowski on the product of n non-homogeneous linear forms.

MSC : 11H31, 11H46, 11J20, 11J37, 52C15.

Keywords : Lattice, Covering, Non-homogeneous, Product of linear forms, Critical determinant, Korkine and Zolotareff reduction, Hermite's constant, Center density.

1 Introduction

Let $L_i = a_{i1}x_1 + \cdots + a_{in}x_n$, $1 \leq i \leq n$ be n real linear forms in n variables x_1, \dots, x_n and having determinant $\Delta = \det(a_{ij}) \neq 0$. The following conjecture is attributed to H. Minkowski:

Conjecture I: *For any given real numbers c_1, \dots, c_n , there exists integers x_1, \dots, x_n such that*

$$|(L_1 + c_1) \cdots (L_n + c_n)| \leq \frac{1}{2^n} |\Delta|. \quad (1.1)$$

Equality is necessary if and only if after a suitable unimodular transformation the linear forms L_i have the form $2c_i x_i$ for $1 \leq i \leq n$.

*The author acknowledges the support of CSIR, India. The paper forms a part of her Ph.D. dissertation accepted by Panjab University, Chandigarh.

This result is known to be true for $n \leq 9$. For a detailed history and the related results, see Bambah et al [1], Gruber [8], Hans-Gill et al [11] and Kathuria and Raka [17].

Minkowski's Conjecture is equivalent to saying that

$$M_n \leq \frac{1}{2^n} |\Delta|,$$

where $M_n = M_n(\Delta)$ is given by

$$M_n = \sup_{L_1, \dots, L_n} \sup_{(c_1, \dots, c_n) \in \mathbb{R}^n} \inf_{(u_1, \dots, u_n) \in \mathbb{Z}^n} \prod_{i=1}^n |L_i(u_1, \dots, u_n) + c_i|.$$

Chebotarev [4] proved the weaker inequality

$$M_n \leq \frac{1}{2^{n/2}} |\Delta|. \quad (1.2)$$

Since then several authors have tried to improve upon this estimate. The bounds have been obtained in the form

$$M_n \leq \frac{1}{\nu_n 2^{n/2}} |\Delta|, \quad (1.3)$$

where $\nu_n > 1$. Clearly $\nu_n \leq 2^{n/2}$ by considering the linear forms $L_i = x_i$ and $c_i = \frac{1}{2}$ for $1 \leq i \leq n$. During 1949-1986, many authors such as Davenport, Woods, Bombieri, Gruber, Skubenko, Andrijasjan, Il'in and Malyshev obtained ν_n for large n . For details, see Gruber and Lekkerkerker [9], Hans-Gill et al [12]. In 1960, Mordell [21] obtained $\nu_n = 4 - 2(2 - 3\sqrt{2}/4)^n - 2^{-n/2}$ for all n . Il'in [15, 16] (1986, 1991) improved Mordell's estimates for $6 \leq n \leq 31$. Hans-Gill et al [12, 14] (2010, 2011) got improvements on the results of Il'in [16] for $9 \leq n \leq 31$. Since recently $\nu_9 = 2^{9/2}$ has been established by the authors [17], we study ν_n for $10 \leq n \leq 33$ and obtain their refined values in this paper.

For sake of comparison, we give results by Mordell [21], Il'in [16], Hans-Gill et al [14] and our improved ν_n in Table I.

We shall follow the Remak-Davenport approach. For the sake of convenience of the reader we give some basic results of this approach.

Minkowski's Conjecture can be restated in the terminology of lattices as : Any lattice Λ of determinant $d(\Lambda)$ in \mathbb{R}^n is a covering lattice for the set

$$S : |x_1 x_2 \dots x_n| \leq \frac{d(\Lambda)}{2^n}.$$

Table I

	Estimates by Mordell	Estimates by Il'in	Estimates by Hans-Gill et al	Our improved Estimates
n	ν_n	ν_n	ν_n	ν_n
10	2.8990614	3.4798928	24.3627506	27.6034811
11	2.9731018	3.5229055	29.2801145	33.4727227
12	3.0405253	3.5502417	32.2801213	39.5919904
13	3.1023558	3.5785628	34.8475153	45.4004068
14	3.1593729	3.6020935	37.8038391	51.2623882
15	3.2121798	3.6111553	40.9051980	57.0037507
16	3.2612520	3.6190753	44.3414913	57.4701963
17	3.3069717	3.6392444	47.2339309	57.6759791
18	3.3496524	3.6617581	46.7645724	57.3887589
19	3.3895562	3.6673429	47.2575897	60.0933912
20	3.4269065	3.6723611	46.8640155	58.4859214
21	3.4618973	3.6769169	46.0522028	56.4257125
22	3.4946990	3.684080	43.6612034	53.9414220
23	3.5254641	3.6863331	37.8802374	50.9884152
24	3.5543297	3.6897821	32.5852958	47.7463213
25	3.5814208	3.6929517	27.8149432	42.3908768
26	3.6068520	3.6958893	23.0801951	38.8656991
27	3.6307288	3.7001150	17.3895105	31.9331584
28	3.6531489	3.7026271	12.9938763	26.1066323
29	3.6742031	3.7049722	9.5796191	19.9625412
30	3.6939760	3.7086731	6.7664335	16.0688443
31	3.7125466	3.7255824	4.7459720	11.2387160
32	3.7299885			8.3258788
33	3.746371			5.4114880

The weaker result (1.3) is equivalent to saying that any lattice Λ of determinant $d(\Lambda)$ in \mathbb{R}^n is a covering lattice for the set

$$S : |x_1 x_2 \dots x_n| \leq \frac{d(\Lambda)}{\nu_n 2^{n/2}}.$$

Define the homogeneous minimum of Λ as

$$m_H(\Lambda) = \inf\{|x_1 x_2 \dots x_n| : X = (x_1, x_2, \dots, x_n) \in \Lambda, X \neq O\}.$$

In 1956, Birch and Swinnerton-Dyer[2] proved

Proposition 1. Suppose that Minkowski Conjecture has been proved for dimensions $1, 2, \dots, n-1$. Then it holds for all lattices Λ in \mathbb{R}^n for which $M_H(\Lambda) = 0$.

C.T. McMullen[20] proved

Proposition 2. If Λ is a lattice in \mathbb{R}^n for $n \geq 3$ with $M_H(\Lambda) \neq 0$ then there

exists an ellipsoid having n linearly independent points of Λ on its boundary and no point of Λ other than O in its interior.

It is well known that using these results, Minkowski's Conjecture would follow from

Conjecture II. If Λ is a lattice in \mathbb{R}^n of determinant 1 and there is a sphere $|X| < R$ which contains no point of Λ other than O in its interior and has n linearly independent points of Λ on its boundary then Λ is a covering lattice for the closed sphere of radius $\sqrt{n/4}$. Equivalently, every closed sphere of radius $\sqrt{n/4}$ lying in \mathbb{R}^n contains a point of Λ .

Woods [24, 25] formulated a conjecture from which Conjecture-II follows immediately. To state Woods' conjecture, we need to introduce some terminology :

Let \mathbb{L} be a lattice in \mathbb{R}^n . By the reduction theory of quadratic forms introduced by Korkine and Zolotareff [19], a cartesian co-ordinate system may be chosen in \mathbb{R}^n in such a way that \mathbb{L} has a basis of the form

$$(A_1, 0, 0, \dots, 0), (a_{2,1}, A_2, 0, \dots, 0), \dots, (a_{n,1}, a_{n,2}, \dots, a_{n,n-1}, A_n),$$

where A_1, A_2, \dots, A_n are all positive and further for each $i = 1, 2, \dots, n$ any two points of the lattice in \mathbb{R}^{n-i+1} with basis

$$(A_i, 0, 0, \dots, 0), (a_{i+1,i}, A_{i+1}, 0, \dots, 0), \dots, (a_{n,i}, a_{n,i+1}, \dots, a_{n,n-1}, A_n)$$

are at a distance atleast A_i apart. Such a basis of \mathbb{L} is called a reduced basis.

Conjecture III (Woods): *If $A_1 A_2 \cdots A_n = 1$ and $A_i \leq A_1$ for each i then any closed sphere in \mathbb{R}^n of radius $\sqrt{n}/2$ contains a point of \mathbb{L} .*

Woods [23, 24, 25] proved this conjecture for $4 \leq n \leq 6$. Hans-Gill et al [10] gave a unified proof of Woods' Conjecture for $n \leq 6$. Hans-Gill et al [12, 14] proved Woods' Conjecture for $n = 7$ and $n = 8$ and thus completed the proof of Minkowski's Conjecture for $n = 7$ and 8. Kathuria and Raka [17] proved Woods Conjecture and hence Minkowski's Conjecture for $n = 9$. With the assumptions as in Conjecture III, a weaker result would be that

If $\omega_n \geq n$, any closed sphere in \mathbb{R}^n of radius $\sqrt{\omega_n}/2$ contains a point of \mathbb{L} .

Hans-Gill et al [12, 14] obtained the estimates ω_n on Woods' Conjecture for $n \geq 9$. As $\omega_9 = 9$ has been established by the authors [17] recently, in this paper we obtain improved estimates ω_n on Woods' Conjecture for $10 \leq n \leq 33$. Together with the following result of Hans-Gill et al. [12], we get improvements of ω_n for $n \geq 34$ also.

Proposition 3. *Let \mathbb{L} be a lattice in \mathbb{R}^n with $A_1 A_2 \cdots A_n = 1$ and $A_i \leq A_1$ for each i . Let $0 < l_n \leq A_n^2 \leq m_n$, where l_n and m_n are real numbers. Then \mathbb{L} is a covering lattice for the sphere $|X| \leq \sqrt{\omega_n}/2$, where ω_n is defined inductively by*

$$\omega_n = \max\{\omega_{n-1} l_n^{-1/l_{n-1}} + l_n, \omega_{n-1} m_n^{-1/m_{n-1}} + m_n\}.$$

Here we prove

Theorem 1. *Let $10 \leq n \leq 33$. If $d(\mathbb{L}) = A_1 \cdots A_n = 1$ and $A_i \leq A_1$ for $i = 2, \dots, n$, then any closed sphere in \mathbb{R}^n of radius $\sqrt{\omega_n}/2$ contains a point of \mathbb{L} , where ω_n are as listed in Table II.*

For the sake of comparison we give results by Hans-Gill et. al [14] and our improved ω_n in Table II.

To deduce the results on the estimates of Minkowski's Conjecture we also need the following generalization of Proposition 1 (see Theorem 3 of [12]; for a proof see [18]):

Proposition 4. *Suppose that we know*

$$M_j \leq \frac{1}{\nu_j 2^{j/2} |\Delta|} \text{ for } 1 \leq j \leq n-1.$$

Let $\nu_n < \min \nu_{k_1} \nu_{k_2} \cdots \nu_{k_s}$, where the minimum is taken over all (k_1, k_2, \dots, k_s) such that $n = k_1 + k_2 + \dots + k_s$, k_i positive integers for all i and $s \geq 2$. Then for all lattices Λ in \mathbb{R}^n with homogeneous minimum $M_H(\Lambda) = 0$, the estimate ν_n holds for Minkowski's Conjecture.

Since by arithmetic-geometric inequality the sphere $\{X \in \mathbb{R}^n : |X| \leq \frac{\sqrt{\omega_n}}{2}\}$ is a subset of $\{X : |x_1 x_2 \cdots x_n| \leq \frac{1}{2^{n/2}} (\frac{\omega_n}{2^n})^{n/2}\}$, Propositions 2 and 4 immediately imply

Theorem 2: *The values of ν_n for the estimates of Minkowski's Conjecture can be taken as $(\frac{2n}{\omega_n})^{n/2}$.*

For $10 \leq n \leq 33$, these values are listed in Table I. In Section 2 we state some preliminary results and in Sections 3-9 we prove Theorem 1 for $10 \leq n \leq 33$.

2 Preliminary Results and Plan of the Proof

Let \mathbb{L} be a lattice in \mathbb{R}^n reduced in the sense of Korkine and Zolotareff. Let $\Delta(S_n)$ denotes the critical determinant of the unit sphere S_n with center O in \mathbb{R}^n i.e.

$$\Delta(S_n) = \inf\{d(\Lambda) : \Lambda \text{ has no point other than } O \text{ in the interior of } S_n\}.$$

Let γ_n be the Hermite's constant i.e. γ_n is the smallest real number such that

for any positive definite quadratic form Q in n variables of determinant D , there exist integers u_1, u_2, \dots, u_n not all zero satisfying

$$Q(u_1, u_2, \dots, u_n) \leq \gamma_n D^{1/n}.$$

It is well known that $\Delta^2(S_n) = \gamma_n^{-n}$. We write $A_i^2 = B_i$.

We state below some preliminary lemmas. Lemmas 1 and 2 are due to Woods [23], Lemma 3 is due to Korkine and Zolotareff [19], and Lemma 4 is due to Pendavingh and Van Zwam [22]. In Lemma 5, the cases $n = 2$ and 3 are classical results of Lagrange and Gauss; $n = 4$ and 5 are due to Korkine and Zolotareff [19] while $n = 6, 7$ and 8 are due to Blichfeldt [3].

Lemma 1. If $2\Delta(S_{n+1})A_1^n \geq d(\mathbb{L})$ then any closed sphere of radius

$$R = A_1(1 - \{A_1^n \Delta(S_{n+1})/d(\mathbb{L})\}^2)^{1/2}$$

in \mathbb{R}^n contains a point of \mathbb{L} .

Lemma 2. For a fixed integer i with $1 \leq i \leq n-1$, denote by \mathbb{L}_1 the lattice in \mathbb{R}^i with reduced basis

$$(A_1, 0, \dots, 0), (a_{2,1}, A_2, 0, \dots, 0), \dots, (a_{i,1}, a_{i,2}, \dots, a_{i,i-1}, A_i)$$

and denote by \mathbb{L}_2 the lattice in \mathbb{R}^{n-i} with reduced basis

$$(A_{i+1}, 0, \dots, 0), (a_{i+2,i+1}, A_{i+2}, 0, \dots, 0), \dots, (a_{n,i+1}, a_{n,i+2}, \dots, a_{n,n-1}, A_n).$$

If any closed sphere in \mathbb{R}^i of radius r_1 contains a point of \mathbb{L}_1 and if any closed sphere in \mathbb{R}^{n-i} of radius r_2 contains a point of \mathbb{L}_2 then any closed sphere in \mathbb{R}^n of radius $(r_1^2 + r_2^2)^{1/2}$ contains a point of \mathbb{L} .

Lemma 3. For all relevant i ,

$$B_{i+1} \geq \frac{3}{4}B_i \text{ and } B_{i+2} \geq \frac{2}{3}B_i. \quad (2.1)$$

Lemma 4. For all relevant i ,

$$B_{i+4} \geq (0.46873)B_i. \quad (2.2)$$

Throughout the paper we shall denote 0.46873 by ε .

Lemma 5. $\Delta(S_n) = \sqrt{3}/2, 1/\sqrt{2}, 1/2, 1/2\sqrt{2}, \sqrt{3}/8, 1/8$ and $1/16$ for $n = 2, 3, 4, 5, 6, 7$ and 8 respectively.

Lemma 6. For any integer s , $1 \leq s \leq n-1$

$$B_1 B_2 \dots B_{s-1} B_s^{n-s+1} \leq \gamma_{n-s+1}^{n-s+1} \text{ and} \quad (2.3)$$

$$B_1 B_2 \dots B_s \leq (\gamma_n^{\frac{1}{n-1}} \gamma_{n-1}^{\frac{1}{n-2}} \dots \gamma_{n-s+1}^{\frac{1}{n-s}})^{n-s}. \quad (2.4)$$

This is Lemma 4 of Hans-Gill et al [12].

Lemma 7.

$$\{(8.5337)^{\frac{1}{5}}\gamma_n^{\frac{1}{n-1}}\gamma_{n-1}^{\frac{1}{n-2}}\dots\gamma_6^{\frac{1}{5}}\}^{-1} \leq B_n \leq \gamma_{n-1}^{\frac{n-1}{n}}. \quad (2.5)$$

This is Lemma 6 of Hans-Gill et al [14].

Remark 1. Let

- δ_n = the best centre density of packings of unit spheres in \mathbb{R}^n ,
- δ_n^* = the best centre density of lattice packings of unit spheres in \mathbb{R}^n .

Then it is known that (see Conway and Sloane [7], page 20)

$$\gamma_n = 4(\delta_n^*)^{\frac{2}{n}} \leq 4(\delta_n)^{\frac{2}{n}}. \quad (2.6)$$

δ_n^* and hence γ_n is known for $n \leq 8$. Also $\gamma_{24} = 4$ has been proved by Cohn and Kumar [6]. Using the bounds on δ_n given by Cohn and Elkies [5] and inequality (2.6) we find bounds on γ_n for $10 \leq n \leq 33$ which we have listed in Table II. Also $\gamma_9 \leq 2.1326324$.

Table II

n	$\gamma_n \leq$	Estimates by Hans-Gill et al ω_n	Our improved Estimates ω_n
10	2.2636302	10.5605061	10.3
11	2.3933470	11.9061976	11.62
12	2.5217871	13.4499927	13
13	2.6492947	15.0562267	14.455765
14	2.7758041	16.6646332	15.955156
15	2.9014777	18.2901579	17.498499
16	3.0263937	19.9204292	19.285
17	3.1506793	21.6026907	21.101
18	3.2743307	23.4831402	22.955
19	3.3974439	25.3234826	24.691
20	3.5200620	27.2255111	26.629
21	3.6422432	29.1638254	28.605
22	3.7640371	31.2142617	30.62
23	3.8854763	33.5354821	32.68
24	4.0065998	35.9050965	34.78
25	4.1274438	38.3201985	37.05
26	4.2480446	40.8449876	39.24
27	4.3684312	43.7039431	41.78
28	4.488631	46.6267624	44.36
29	4.6086676	49.6305176	47.18
30	4.7285667	52.8194566	49.86
31	4.8483483	56.0735184	53.04
32	4.9680344		56.06
33	5.0876409		59.58

We assume that Theorem 1 is false and derive a contradiction. Let \mathbb{L} be a lattice satisfying the hypothesis of the conjecture. Suppose that there exists a closed sphere of radius $\sqrt{\omega_n}/2$ in \mathbb{R}^n that contains no point of \mathbb{L} in \mathbb{R}^n . Since $B_i = A_i^2$ and $d(\mathbb{L}) = 1$, we have $B_1 B_2 \dots B_n = 1$.

We give some examples of inequalities that arise. Let \mathbb{L}_1 be a lattice in \mathbb{R}^4 with basis $(A_1, 0, 0, 0)$, $(a_{2,1}, A_2, 0, 0)$, $(a_{3,1}, a_{3,2}, A_3, 0)$, $(a_{4,1}, a_{4,2}, a_{4,3}, A_4)$, and \mathbb{L}_i for $2 \leq i \leq n$ be lattices in \mathbb{R}^1 with basis (A_{i+3}) . Applying Lemma 2 repeatedly and using Lemma 1 we see that if $2\Delta(S_5)A_1^4 \geq A_1 A_2 A_3 A_4$ then any closed sphere of radius

$$\left(A_1^2 - \frac{A_1^{10} \Delta(S_5)^2}{A_1^2 A_2^2 A_3^2 A_4^2} + \frac{1}{4} A_5^2 + \dots + \frac{1}{4} A_n^2 \right)^{1/2}$$

contains a point of \mathbb{L} . By the initial hypothesis this radius exceeds $\sqrt{\omega_n}/2$. Since $\Delta(S_5) = 1/2\sqrt{2}$ and $B_1 B_2 \dots B_n = 1$, this results in the conditional inequality : if $B_1^4 B_5 B_6 \dots B_n \geq 2$ then

$$4B_1 - \frac{1}{2} B_1^5 B_5 B_6 \dots B_n + B_5 + B_6 + \dots + B_n > \omega_n. \quad (2.7)$$

We call this inequality $(4, 1, \dots, 1)$, since it corresponds to the ordered partition $(4, 1, \dots, 1)$ of n for the purpose of applying Lemma 2. Similarly the conditional inequality $(1, \dots, 1, 2, 1, \dots, 1)$ corresponding to the ordered partition $(1, \dots, 1, 2, 1, \dots, 1)$ is : if $2B_i \geq B_{i+1}$ then

$$B_1 + \dots + B_{i-1} + 4B_i - \frac{2B_i^2}{B_{i+1}} + B_{i+2} + \dots + B_n > \omega_n. \quad (2.8)$$

Since $4B_i - \frac{2B_i^2}{B_{i+1}} \leq 2B_{i+1}$, (2.8) gives

$$B_1 + \dots + B_{i-1} + 2B_{i+1} + B_{i+2} + \dots + B_n > \omega_n. \quad (2.9)$$

One may remark here that the condition $2B_i \geq B_{i+1}$ is necessary only if we want to use inequality (2.8), but it is not necessary if we want to use the weaker inequality (2.9). This is so because if $2B_i < B_{i+1}$, using the partition $(1, 1)$ in place of (2) for the relevant part, we get the upper bound $B_i + B_{i+1}$ which is clearly less than $2B_{i+1}$. We shall call inequalities of type (2.9) as *weak* inequalities and denote it by $(1, \dots, 1, 2, 1, \dots, 1)_w$.

If $(\lambda_1, \lambda_2, \dots, \lambda_s)$ is an ordered partition of n , then the conditional inequality arising from it, by using Lemmas 1 and 2, is also denoted by $(\lambda_1, \lambda_2, \dots, \lambda_s)$. If the conditions in an inequality $(\lambda_1, \lambda_2, \dots, \lambda_s)$ are satisfied then we say that $(\lambda_1, \lambda_2, \dots, \lambda_s)$ holds.

Sometimes, instead of Lemma 2, we are able to use induction. The use

of this is indicated by putting $(*)$ on the corresponding part of the partition. For example, if for $n = 10$, B_5 is larger than each of B_6, B_7, \dots, B_{10} , and if $\frac{B_1^3}{B_2 B_3 B_4} > 2$, the inequality $(4, 6^*)$ gives

$$4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 6(B_1 B_2 B_3 B_4)^{-1/6} > \omega_{10}. \quad (2.10)$$

In particular the inequality $((n-1)^*, 1)$ always holds. This can be written as

$$\omega_{n-1}(B_n)^{\frac{-1}{(n-1)}} + B_n > \omega_n. \quad (2.11)$$

Also we have $B_1 \geq 1$, because if $B_1 < 1$, then $B_i \leq B_1 < 1$ for each i contradicting $B_1 B_2 \dots B_n = 1$.

Using the upper bounds on γ_n and the inequality (2.5), we obtain numerical lower and upper bounds on B_n , which we denote by l_n and m_n respectively. We use the approach of Hans-Gill et al [14], but our method of dealing with is somewhat different. In Sections 3-5 we give proof of Theorem 1 for $n = 10, 11$ and 12 respectively. The proof of these cases is based on the truncation of the interval $[l_n, m_n]$ from both the sides. In Sections 6-8 we give proof of Theorem 1 for $n = 13, 14$ and 15 . The proof of these cases is based on the truncation of the interval $[l_n, m_n]$ from one side only. (Truncation from both the sides makes the proof very complicated and it does not give any significant improvement on ω_n .) For $16 \leq n \leq 33$ we use the inequalities in somewhat different way and this is discussed in Section 9.

In this paper we need to maximize or minimize frequently functions of several variables. When we say that a given function of several variables in x, y, \dots is an increasing/decreasing function of x, y, \dots , it means that the concerned property holds when function is considered as a function of one variable at a time, all other variables being fixed.

3 Proof of Theorem 1 for $n = 10$

Here we have $\omega_{10} = 10.3$, $B_1 \leq \gamma_{10} < 2.2636302$. Using (2.5), we have $l_{10} = 0.4007 < B_{10} < 1.9770808 = m_{10}$.

The inequality $(9^*, 1)$ gives $9(B_{10})^{\frac{-1}{9}} + B_{10} < 10.3$. But for $0.4398 \leq B_{10} \leq 1.9378$, this inequality is not true. Hence we must have either $B_{10} < 0.4398$ or $B_{10} > 1.9378$.

We will deal with the two cases $0.4007 < B_{10} < 0.4398$ and $1.9378 < B_{10} < 1.9770808$ separately:

3.1 $0.4007 < B_{10} < 0.4398$

Using (2.1),(2.2) we have:

$$\begin{cases} B_9 \leq \frac{4}{3}B_{10} < 0.5864, & B_8 \leq \frac{3}{2}B_{10} < 0.6597, & B_7 \leq 2B_{10} < 0.8796, \\ B_6 \leq \frac{B_{10}}{\varepsilon} < 0.9383, & B_5 \leq \frac{4}{3}\frac{B_{10}}{\varepsilon} < 1.2511, & B_4 \leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 1.4075, \\ B_3 \leq \frac{2B_{10}}{\varepsilon} < 1.8766, & B_2 \leq \frac{B_{10}}{(\varepsilon)^2} < 2.0018 \end{cases} . \quad (3.1)$$

Claim(i) $B_2 > 1.7046$

The inequality $(2, 2, 2, 2, 2)_w$ gives $2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} > 10.3$. Using (3.1), we find that this inequality is not true for $B_2 \leq 1.7046$. Hence we must have $B_2 > 1.7046$.

Claim(ii) $B_2 < 1.8815$

Suppose $B_2 \geq 1.8815$, then using (3.1) and that $B_6 \geq \varepsilon B_2$ we find that $\frac{B_3^3}{B_3 B_4 B_5} > 2$ and $\frac{B_6^3}{B_7 B_8 B_9} > 2$. So the inequality $(1, 4, 4, 1)$ holds, i.e. $B_1 + 4B_2 - \frac{1}{2}\frac{B_2^4}{B_3 B_4 B_5} + 4B_6 - \frac{1}{2}\frac{B_6^4}{B_7 B_8 B_9} + B_{10} > 10.3$. Applying AM-GM inequality we get $B_1 + 4B_2 + 4B_6 + B_{10} - \sqrt{B_2^5 B_6^5 B_1 B_{10}} > 10.3$. Now since $\varepsilon^2 B_2 \leq B_{10} < 0.4398$, $B_6 \geq \varepsilon B_2$, $B_1 \geq B_2$ and $B_2 \geq 1.8815$, we find that the left side is a decreasing function of B_{10} and B_6 . So replacing B_{10} by $\varepsilon^2 B_2$ and B_6 by εB_2 we get $\phi_1 = B_1 + (4 + 4\varepsilon + \varepsilon^2)B_2 - \sqrt{(\varepsilon)^7 B_2^{11} B_1} > 10.3$. Now the left side is a decreasing function of B_2 , so replacing B_2 by 1.8815 we find that $\phi_1 < 10.3$ for $1 < B_1 < 2.2636302$, a contradiction. Hence we must have $B_2 < 1.8815$.

Claim(iii) $B_3 < 1.5652$

Suppose $B_3 \geq 1.5652$. From (3.1) we have $B_4 B_5 B_6 < 1.6524$ and $B_8 B_9 B_{10} < 0.1702$, so we find that $\frac{B_3^3}{B_4 B_5 B_6} > 2$ and $\frac{B_7^3}{B_8 B_9 B_{10}} \geq \frac{(\varepsilon B_3)^3}{B_8 B_9 B_{10}} > 2$, for $B_3 > 1.49$.

Applying AM-GM to inequality (2, 4, 4) we get $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 + 4B_7 - \sqrt{B_3^5 B_7^5 B_1 B_2} > 10.3$. Since $B_1 \geq B_2 > 1.7046$, $B_7 \geq \varepsilon B_3$ and $B_3 \geq 1.5652$, we find that left side is a decreasing function of B_1 and B_7 . So we replace B_1 by B_2 , B_7 by εB_3 and get that $\phi_2 = 2B_2 + 4(1 + \varepsilon)B_3 - \sqrt{(\varepsilon)^5 B_3^{10} B_2^2} > 10.3$. But left side is a decreasing function of B_3 , so replacing B_3 by 1.5652 we find that $\phi_2 < 10.3$ for $1.7046 < B_2 < 1.8815$, a contradiction. Hence we must have $B_3 < 1.5652$.

Claim(iv) $B_1 > 1.9378$

Suppose $B_1 \leq 1.9378$. Using (3.1) and that $B_3 < 1.5652$, $B_2 > 1.7046$, we find that B_2 is larger than each of B_3, B_4, \dots, B_{10} . So the inequality $(1, 9^*)$ holds. This gives $B_1 + 9(B_1)^{-1/9} > 10.3$, which is not true for $B_1 \leq 1.9378$. So we must have $B_1 > 1.9378$.

Claim(v) $B_3 < 1.5485$

Suppose $B_3 \geq 1.5485$. We proceed as in Claim(iii) and replace B_1 by

1.9378 and B_7 by εB_3 to get that $\phi_3 = 4(1.9378) - \frac{2(1.9378)^2}{B_2} + 4(1 + \varepsilon)B_3 - \sqrt{(\varepsilon)^5(1.9378)B_3^{10}B_2} > 10.3$. One easily checks that $\phi_3 < 10.3$ for $1.5485 \leq B_3 < 1.5652$ and $1.7046 < B_2 < 1.8815$. Hence we have $B_3 < 1.5485$.

Claim(vi) $B_1 < 2.0187$

Suppose $B_1 \geq 2.0187$. Using (3.1) and Claims (ii), (v) we have $B_2B_3B_4 < 4.11$. Therefore $\frac{B_1^3}{B_2B_3B_4} > 2$. As $B_5 \geq \varepsilon B_1 > 0.9462$, we see using (3.1) that B_5 is larger than each of B_6, B_7, \dots, B_{10} . Hence the inequality $(4, 6^*)$ holds. This gives $\phi_4 = 4B_1 - \frac{1}{2} \frac{B_1^4}{B_2B_3B_4} + 6(B_1B_2B_3B_4)^{-1/6} > 10.3$. Left side is an increasing function of $B_2B_3B_4$ and decreasing function of B_1 . So we can replace $B_2B_3B_4$ by 4.11 and B_1 by 2.0187 to find $\phi_4 < 10.3$, a contradiction. Hence we have $B_1 < 2.0187$.

Claim(vii) $B_4 < 1.337$

Suppose $B_4 \geq 1.337$, then using (3.1) we get $\frac{B_4^3}{B_5B_6B_7} > 2$. Applying AM-GM to inequality $(1, 2, 4, 2, 1)$ we have $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 + 4B_8 + B_{10} - 2\sqrt{B_4^5B_8^3B_1B_2B_3B_{10}} > 10.3$. Since $B_2 > 1.7046$, $B_3 \geq \frac{3}{4}B_2$, $B_4 \geq 1.337$, $B_8 \geq \varepsilon B_4$ and $B_{10} \geq \frac{2\varepsilon}{3}B_4$, we find that left side is a decreasing function of B_2 , B_8 and B_{10} . So we can replace B_2 by 1.7046; B_8 by εB_4 and B_{10} by $\frac{2\varepsilon}{3}B_4$ to get $\phi_5 = B_1 + 4(1.7046) - \frac{2(1.7046)^2}{B_3} + (4 + 4\varepsilon + \frac{2\varepsilon}{3})B_4 - 2\sqrt{\frac{2}{3}(\varepsilon)^4(1.7046)B_4^9B_1B_3} > 10.3$. Now left side is a decreasing function of B_4 , replacing B_4 by 1.337, we find that $\phi_5 < 10.3$ for $1 < B_1 < 2.0187$ and $1 < B_3 < 1.5485$, a contradiction. Hence we have $B_4 < 1.337$.

Claim(viii) $B_5 < 1.1492$

Suppose $B_5 \geq 1.1492$. Using (3.1), we get $B_6B_7B_8 < 0.5445$. Therefore $\frac{B_5^3}{B_6B_7B_8} > 2$. Also using (2.1), (2.2), $2B_9 \geq 2(\varepsilon B_5) > 1.077 > B_{10}$. So the inequality $(4^*, 4, 2)$ holds, i.e. $4(\frac{1}{B_5B_6B_7B_8B_9B_{10}})^{\frac{1}{4}} + 4B_5 - \frac{1}{2} \frac{B_5^4}{B_6B_7B_8} + 4B_9 - \frac{2B_9^2}{B_{10}} > 10.3$. Now left side is a decreasing function of B_5 and B_9 . So we replace B_5 by 1.1492 and B_9 by 1.1492ε and get that $\phi_6(x, B_{10}) = 4(\frac{1}{(\varepsilon)(1.1492)^2xB_{10}})^{\frac{1}{4}} + 4(1 + \varepsilon)(1.1492) - \frac{1}{2} \frac{(1.1492)^4}{x} - \frac{2(1.1492\varepsilon)^2}{B_{10}} > 10.3$, where $x = B_6B_7B_8$. Using (2.1), (2.2) we have $x = B_6B_7B_8 \geq \frac{B_5^3}{4} \geq \frac{(1.1492)^3}{4}$ and $B_{10} \geq \frac{3\varepsilon}{4}B_5 \geq \frac{3\varepsilon}{4}(1.1492)$. It can be verified that $\phi_6(x, B_{10}) < 10.3$ for $\frac{(1.1492)^3}{4} \leq x < 0.5445$ and $\frac{3\varepsilon}{4}(1.1492) \leq B_{10} < 0.4398$, giving thereby a contradiction. Hence we must have $B_5 < 1.1492$.

Claim(ix) $B_2 < 1.766$.

Suppose $B_2 \geq 1.766$. We have $B_3B_4B_5 < 2.3793$. So $\frac{B_2^3}{B_3B_4B_5} > 2$. Also $B_6 \geq \varepsilon B_2 > 0.8277$. Therefore B_6 is larger than each of B_7, B_8, B_9, B_{10} . Hence the inequality $(1, 4, 5^*)$ holds. This gives $B_1 + 4B_2 - \frac{1}{2} \frac{B_2^4}{B_3B_4B_5} + 5(\frac{1}{B_1B_2B_3B_4B_5})^{\frac{1}{5}} > 10.3$. Left side is an increasing function of $B_3B_4B_5$, a decreasing function of B_2 and an increasing function of B_1 . One easily

checks that this inequality is not true for $B_1 < 2.0187$; $B_2 \geq 1.766$ and $B_3B_4B_5 < 2.3793$. Hence we have $B_2 < 1.766$.

Final contradiction

As $2(B_2 + B_4 + B_6 + B_8 + B_{10}) < 2(1.766 + 1.337 + 0.9383 + 0.6597 + 0.4398) < 10.3$, the weak inequality $(2, 2, 2, 2, 2)_w$ gives a contradiction.

3.2 $1.9378 < B_{10} < 1.9770808$

Here $B_1 \geq B_{10} > 1.9378$. And $B_2 = (B_1B_3 \cdots B_{10})^{-1}$
 $\leq (B_1 \cdot \frac{3}{4}B_2 \cdot \frac{2}{3}B_2 \cdot \frac{1}{2}B_2 \cdot \varepsilon B_2 \cdot \frac{3\varepsilon}{4}B_2 \cdot \frac{2\varepsilon}{3}B_2 \cdot \frac{\varepsilon}{2}B_2 \cdot B_{10})^{-1} = (\frac{1}{16}\varepsilon^4 B_2^7 B_1 B_{10})^{-1}$,
which implies $(B_2)^8 \leq (\frac{1}{16}\varepsilon^4 (1.9378)^2)^{-1}$, i.e. $B_2 < 1.75076$.

Similarly

$$\begin{aligned} B_3 &= (B_1B_2B_4 \cdots B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^3 B_3^6 B_1^2 B_{10})^{-1}; \\ B_4 &= (B_1B_2B_3B_5 \cdots B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^2 B_4^5 B_1^3 B_{10})^{-1}; \\ B_6 &= (B_1 \cdots B_5B_7B_8B_9B_{10})^{-1} \leq (\frac{1}{16}\varepsilon B_6^3 B_1^5 B_{10})^{-1}; \\ B_8 &= (B_1 \cdots B_7B_9B_{10})^{-1} \leq (\frac{3}{32}\varepsilon^3 B_8 B_1^7 B_{10})^{-1}. \end{aligned}$$

These respectively give $B_3 < 1.46138$, $B_4 < 1.22883$, $B_6 < 0.896058$ and $B_8 < 0.721763$. So we have $B_1^4 B_5 B_6 B_7 B_8 B_9 B_{10} = \frac{B_1^3}{B_2 B_3 B_4} > 2$. Also $2B_5 \geq 2(\varepsilon B_1) > 1.8166 > B_6$ and $2B_7 \geq 2(\frac{2\varepsilon}{3}B_1) > B_8$.

Applying AM-GM to inequality $(4, 2, 2, 1, 1)$ we have $4B_1 + 4B_5 + 4B_7 + B_9 + B_{10} - 3(2B_1^5 B_5^3 B_7^3 B_9 B_{10})^{\frac{1}{3}} > 10.3$. We find that left side is a decreasing function of B_7 and B_5 , so can replace B_7 by $\frac{2}{3}\varepsilon B_1$ and B_5 by εB_1 ; then it is a decreasing function of B_1 , so replacing B_1 by B_{10} we have $4(1 + \varepsilon + \frac{2}{3}\varepsilon)B_{10} + B_9 + B_{10} - 2^{\frac{4}{3}}(\varepsilon)^2(B_{10})^4(B_9)^{\frac{1}{3}} > 10.3$, which is not true for $(1.9378)\varepsilon^2 < B_9 \leq B_1 < 2.2636302$ and $1.9378 < B_{10} < 1.9770808$. Hence we get a contradiction. \square

4 Proof of Theorem 1 for $n = 11$

Here we have $\omega_{11} = 11.62$, $B_1 \leq \gamma_{11} < 2.393347$. Using (2.5), we have $l_{11} = 0.3673 < B_{11} < 2.1016019 = m_{11}$.

The inequality $(10^*, 1)$ gives $10.3(B_{11})^{\frac{-1}{10}} + B_{11} > 11.62$. But for $0.4409 \leq B_{11} \leq 2.018$ this inequality is not true. So we must have either $B_{11} < 0.4409$ or $B_{11} > 2.018$.

4.1 $0.3673 < B_{11} < 0.4409$

Claim(i) $B_{10} < 0.4692$

Suppose $B_{10} \geq 0.4692$, then $2B_{10} > B_{11}$, so $(9^*, 2)$ holds, i.e. $9(\frac{1}{B_{10}B_{11}})^{\frac{1}{9}} + 4B_{10} - \frac{2B_{10}^2}{B_{11}} > 11.62$. As left side is a decreasing function of B_{10} , we can

replace B_{10} by 0.4692 and find that it is not true for $0.3673 < B_{11} < 0.4409$. Hence we must have $B_{10} < 0.4692$.

Using (2.1),(2.2) we have:

$$\begin{aligned} B_9 &\leq \frac{4}{3}B_{10} < 0.6256, & B_8 &\leq \frac{3}{2}B_{10} < 0.7038, & B_7 &\leq \frac{B_{11}}{\varepsilon} < 0.94063, \\ B_6 &\leq \frac{B_{10}}{\varepsilon} < 1.0011, & B_5 &\leq \frac{4}{3}\frac{B_{10}}{\varepsilon} < 1.3347, & B_4 &\leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 1.50151, \\ B_3 &\leq \frac{B_{11}}{\varepsilon^2} < 2.0068, & B_2 &\leq \frac{B_{10}}{\varepsilon^2} < 2.13557. \end{aligned} \quad (4.1)$$

Claim(ii) $B_2 > 1.913$

The inequality $(2, 2, 2, 2, 2, 1)_w$ gives $2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} + B_{11} > 11.62$. Using (4.1) we find that this inequality is not true for $B_2 \leq 1.913$. So we must have $B_2 > 1.913$.

Claim(iii) $B_3 < 1.761$

Suppose $B_3 \geq 1.761$, then we have $\frac{B_3^3}{B_4B_5B_6} > 2$ and $\frac{B_7^3}{B_8B_9B_{10}} > \frac{(\varepsilon B_3)^3}{B_8B_9B_{10}} > 2$. Applying AM-GM to the inequality $(2, 4, 4, 1)$ we get $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 + 4B_7 + B_{11} - \sqrt{B_3^5B_7^5B_1B_2B_{11}} > 11.62$. One easily finds that it is not true for $B_1 \geq B_2 > 1.913$, $B_3 \geq 1.761$, $B_7 \geq \varepsilon B_3$, $B_{11} \geq \varepsilon^2 B_3$, $1.913 < B_2 < 2.13557$ and $1.761 \leq B_3 < 2.0068$. Hence we must have $B_3 < 1.761$.

Claim(iv) $B_1 < 2.2436$

Suppose $B_1 \geq 2.2436$. As $B_2B_3B_4 < 2.13557 \times 1.761 \times 1.50151 < 5.6468$, we have $\frac{B_1^3}{B_2B_3B_4} > 2$. Also $B_5 \geq \varepsilon B_1 > 1.051$, so B_5 is larger than each of B_6, B_7, \dots, B_{11} . Hence the inequality $(4, 7^*)$ holds. This gives $4B_1 - \frac{1}{2}\frac{B_1^4}{B_2B_3B_4} + 7(\frac{1}{B_1B_2B_3B_4})^{\frac{1}{7}} > 11.62$. Left side is an increasing function of $B_2B_3B_4$ and decreasing function of B_1 . One easily checks that the inequality is not true for $B_2B_3B_4 < 5.6468$ and $B_1 \geq 2.2436$. Hence we have $B_1 < 2.2436$.

Claim(v) $B_4 < 1.4465$ and $B_2 > 1.9686$

Suppose $B_4 \geq 1.4465$. We have $B_5B_6B_7 < 1.2569$ and $B_9B_{10}B_{11} < 0.1295$. Therefore for $B_4 > 1.36$, we have $\frac{B_4^3}{B_5B_6B_7} > 2$ and $\frac{B_8^3}{B_9B_{10}B_{11}} > \frac{(\varepsilon B_4)^3}{B_9B_{10}B_{11}} > 2$. So the inequality $(1, 2, 4, 4)$ holds. Applying AM-GM to inequality $(1, 2, 4, 4)$, we get $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 + 4B_8 - \sqrt{B_4^5B_8^5B_1B_2B_3} > 11.62$. A simple calculation shows that this is not true for $B_1 \geq B_2 > 1.913$, $B_4 \geq 1.4465$, $B_8 \geq \varepsilon B_4$, $B_4 \geq 1.4465$, $B_1 < 2.2436$ and $B_3 < 1.761$. Hence we have $B_4 < 1.4465$.

Further if $B_2 \leq 1.9686$, then $2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} + B_{11} < 11.62$. So the inequality $(2, 2, 2, 2, 2, 1)_w$ gives a contradiction.

Claim(vi) $B_4 < 1.4265$ and $B_2 > 1.9888$

Suppose $B_4 \geq 1.4265$. We proceed as in Claim(v) and get a contradiction with improved bounds on B_2 and B_4 .

Claim(vii) $B_1 < 2.2056$

Suppose $B_1 \geq 2.2056$. We proceed as in Claim(iv) and get a contradiction with improved bounds on B_1 , B_2 and B_4 .

Claim(viii) $B_2 < 2.025$

Suppose $B_2 \geq 2.025$. As $B_3B_4B_5 < 1.761 \times 1.4265 \times 1.3347 < 3.3529$, we have $\frac{B_2^3}{B_3B_4B_5} > 2$. Also $B_6 \geq \varepsilon B_2 > 0.9491$, so B_6 is larger than each of B_7, B_8, \dots, B_{11} . Hence the inequality $(1, 4, 6^*)$ holds, i.e. $B_1 + 4B_2 - \frac{1}{2} \frac{B_2^4}{B_3B_4B_5} + 6(\frac{1}{B_1B_2B_3B_4B_5})^{\frac{1}{6}} > 11.62$. Left side is an increasing function of $B_3B_4B_5$, a decreasing function of B_2 and an increasing function of B_1 . One easily checks that this inequality is not true for $B_1 < 2.2056$; $B_2 \geq 2.025$ and $B_3B_4B_5 < 3.3529$. Hence we have $B_2 < 2.025$.

Claim(ix) $B_1 < 2.1669$

Suppose $B_1 \geq 2.1669$. We proceed as in Claim(iv) and get a contradiction with improved bounds on B_1 , B_2 and B_4 .

Claim(x) $B_4 < 1.403$ and $B_2 > 2.012$

Suppose $B_4 \geq 1.403$. We proceed as in Claim(v) and get a contradiction with improved bounds on B_2 and B_4 .

Final Contradiction:

As now $B_3B_4B_5 < 1.761 \times 1.403 \times 1.3347 < 3.2977$, we have $\frac{B_2^3}{B_3B_4B_5} > 2$ for $B_2 > 2.012$. Also $B_6 \geq \varepsilon B_2 > 0.943 >$ each of B_7, B_8, \dots, B_{11} . Hence the inequality $(1, 4, 6^*)$ holds. Proceeding as in Claim(viii) we find that this inequality is not true for $B_1 < 2.1669$; $B_2 > 2.012$ and $B_3B_4B_5 < 3.2977$, giving thereby a contradiction.

4.2 $2.018 < B_{11} < 2.1016019$

Here $B_1 \geq B_{11} > 2.018$. Therefore using (2.1),(2.2) we have

$$\begin{aligned} B_{10} &= (B_1 \cdots B_9 B_{11})^{-1} \\ &\leq (B_1 \cdot \frac{3}{4} B_1 \cdot \frac{2}{3} B_1 \cdot \frac{1}{2} B_1 \cdot \varepsilon B_1 \cdot \frac{3}{4} \varepsilon B_1 \cdot \frac{2}{3} \varepsilon B_1 \cdot \frac{1}{2} \varepsilon B_1 \cdot \varepsilon^2 B_1 \cdot B_{11})^{-1} \\ &= (\frac{1}{16} \varepsilon^6 B_1^9 B_{11})^{-1} < (\frac{1}{16} \varepsilon^6 (2.018)^{10})^{-1} < 1.34702. \end{aligned}$$

Similarly

$$B_4 = (B_1 B_2 B_3 B_5 \cdots B_{11})^{-1} \leq (\frac{1}{16} \varepsilon^3 B_4^6 B_1^3 B_{11})^{-1}, \text{ which gives } B_4 < 1.37661.$$

Claim(i) $B_{10} < 0.4402$

The inequality $(9^*, 1, 1)$ gives $9(\frac{1}{B_{10}B_{11}})^{\frac{1}{9}} + B_{10} + B_{11} > 11.62$. But this inequality is not true for $0.4402 \leq B_{10} < 1.34702$ and $2.018 < B_{11} < 2.1016019$. Hence we must have $B_{10} < 0.4402$.

Now we have $B_9 \leq \frac{4}{3} B_{10} < 0.58694$, $B_8 \leq \frac{3}{2} B_{10} < 0.6603$, $B_7 \leq 2B_{10} < 0.8804$ and $B_6 \leq \frac{B_{10}}{\varepsilon} < 0.93914$.

Claim(ii) $B_7 < 0.768$

Suppose $B_7 \geq 0.768$. Then $\frac{B_7^3}{B_8 B_9 B_{10}} > 2$, so $(6^*, 4, 1)$ holds. This gives $\phi_7(x) = 6(x)^{1/6} + 4B_7 - \frac{1}{2}B_7^5 B_{11}x + B_{11} > 11.62$, where $x = B_1 B_2 \dots B_6$. The function $\phi_7(x)$ has its maximum value at $x = (\frac{2}{B_7^5 B_{11}})^{6/5}$. Therefore $\phi_7(x) \leq \phi_7((\frac{2}{B_7^5 B_{11}})^{6/5})$, which is less than 11.62 for $0.768 \leq B_7 < 0.8804$ and $2.018 < B_{11} < 2.1016019$. This gives a contradiction.

Now $B_5 \leq \frac{3}{2}B_7 < 1.1521$ and $B_3 \leq \frac{B_7}{\varepsilon} < 1.6385$.

Claim(iii) $B_2 < 1.795$

Suppose $B_2 \geq 1.795$, then $\frac{B_2^3}{B_3 B_4 B_5} > 2$ and $\frac{B_6^3}{B_7 B_8 B_9} > 2$. Applying AM-GM to the inequality $(1, 4, 4, 1, 1)$ we get $B_1 + 4B_2 + 4B_6 + B_{10} + B_{11} - \sqrt{B_2^5 B_6^5 B_1 B_{10} B_{11}} > 11.62$. We find that left side is a decreasing function of B_6 , so we first replace B_6 by εB_2 ; then it is a decreasing function of B_2 , so we replace B_2 by 1.795 and get that $\phi_8(B_{11}) = B_1 + 4(1 + \varepsilon)(1.795) + B_{10} + B_{11} - \sqrt{(\varepsilon)^5 (1.795)^{10} B_1 B_{10} B_{11}} > 11.62$. Now $\phi_8'(B_{11}) > 0$, so $\phi_8(B_{11}) < \max\{\phi_8(2.018), \phi_8(2.1016019)\}$, which can be verified to be at most 11.62 for $(\varepsilon)^2 (1.795) \leq B_{10} < 0.4402$ and $2.018 < B_1 < 2.393347$, giving thereby a contradiction.

Claim(iv) $B_5 < 0.98392$

Suppose $B_5 \geq 0.98392$. We have $\frac{B_1^3}{B_2 B_3 B_4} > 2$ and $\frac{B_5^3}{B_6 B_7 B_8} > 2$. Also $2B_9 \geq 2(\varepsilon B_5) > B_{10}$. Applying AM-GM to the inequality $(4, 4, 2, 1)$ we get $4B_1 + 4B_5 + 4B_9 - \frac{2B_9^2}{B_{10}} + B_{11} - \sqrt{B_1^5 B_5^5 B_9 B_{10} B_{11}} > 11.62$. One can easily check that left side is a decreasing function of B_9 and B_1 so we can replace B_9 by εB_5 and B_1 by B_{11} to get $\phi_9 = 5B_{11} + 4(1 + \varepsilon)B_5 - \frac{2(\varepsilon B_5)^2}{B_{10}} - \sqrt{\varepsilon B_{11}^6 B_5^6 B_{10}} > 11.62$. Now the left side is a decreasing function of B_5 , so replacing B_5 by 0.98392 we see that $\phi_9 < 11.62$ for $\frac{3\varepsilon}{4}(0.98392) < B_{10} < 0.4409$ and $2.018 < B_{11} < 2.1016019$, a contradiction.

Final Contradiction:

As in Claim(iv), we have $\frac{B_1^3}{B_2 B_3 B_4} > 2$. Also $B_5 \geq \varepsilon B_1 > 0.9458 >$ each of B_6, B_7, \dots, B_{10} . Therefore the inequality $(4, 6^*, 1)$ holds, i.e. $\phi_{10} = 4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 6(\frac{1}{B_1 B_2 B_3 B_4 B_{11}})^{\frac{1}{6}} + B_{11} > 11.62$. Left side is an increasing function of $B_2 B_3 B_4$ and B_{11} and decreasing function of B_1 . Using $B_5 < 0.98392$, we have $B_3 \leq \frac{3}{2}B_5 < 1.47588$ and $B_4 \leq \frac{4}{3}B_5 < 1.311894$. One easily checks that $\phi_{10} < 11.62$ for $B_2 B_3 B_4 < 1.795 \times 1.47588 \times 1.311894$, $B_{11} < 2.1016019$ and $B_1 \geq 2.018$. Hence we have a contradiction. \square

5 Proof of Theorem 1 for $n = 12$

Here we have $\omega_{12} = 13$, $B_1 \leq \gamma_{12} < 2.5217871$. Using (2.5), we have $l_{12} = 0.3376 < B_{12} < 2.2254706 = m_{12}$ and using (2.3) we have $B_1 B_2^{11} \leq \gamma_{11}^{11}$, i.e. $B_2 \leq \gamma_{11}^{\frac{11}{12}} < 2.2254706$.

The inequality $(11^*, 1)$ gives $11.62(B_{12})^{-\frac{1}{11}} + B_{12} > 13$. But this is not true for $0.4165 \leq B_{12} \leq 2.17$. So we must have either $B_{12} < 0.4165$ or $B_{12} > 2.17$.

5.1 $0.3376 < B_{12} < 0.4165$

Claim(i) $B_{11} < 0.459$

Suppose $B_{11} \geq 0.459$, then $B_{12} \geq \frac{3}{4}B_{11} > 0.34425$ and $2B_{11} > B_{12}$, so $(10^*, 2)$ holds, i.e. $\phi_{11} = 10.3(\frac{1}{B_{11}B_{12}})^{\frac{1}{10}} + 4B_{11} - \frac{2B_{11}^2}{B_{12}} > 13$. Left side is a decreasing function of B_{11} , so we can replace B_{11} by 0.459 to find that $\phi_{11} < 13$ for $0.34425 < B_{12} < 0.4165$, a contradiction. Hence we have $B_{11} < 0.459$.

Claim(ii) $B_{10} < 0.5432$

Suppose $B_{10} \geq 0.5432$. From (2.1), $B_{11}B_{12} \geq \frac{1}{2}B_{10}^2$ and $B_{10} \leq \frac{3}{2}B_{12}$. Therefore $\frac{1}{2}(0.5432)^2 \leq B_{11}B_{12} < 0.1912$ and $B_{10}^2 > B_{11}B_{12}$, so the inequality $(9^*, 3)$ holds, i.e. $9(\frac{1}{B_{10}B_{11}B_{12}})^{\frac{1}{9}} + 4B_{10} - \frac{B_{10}^3}{B_{11}B_{12}} > 13$. One easily checks that it is not true noting that left side is a decreasing function of B_{10} . Hence we must have $B_{10} < 0.5432$.

Claim(iii) $B_9 < 0.6655$

Suppose $B_9 \geq 0.6655$, then $\frac{B_9^3}{B_{10}B_{11}B_{12}} > 2$. So the inequality $(8^*, 4)$ holds. This gives $\phi_{12}(x) = 8(x)^{1/8} + 4B_9 - \frac{1}{2}B_9^5x > 13$, where $x = B_1B_2 \dots B_8$. The function $\phi_{12}(x)$ has its maximum value at $x = (\frac{2}{B_9^5})^{\frac{8}{7}}$, so $\phi_{12}(x) < \phi_{12}((\frac{2}{B_9^5})^{\frac{8}{7}}) < 13$ for $0.6655 \leq B_9 \leq \frac{3}{2}B_{11} < 0.6885$. This gives a contradiction.

Using (2.1), (2.2) we have:

$$\begin{aligned} B_8 &\leq \frac{3}{2}B_{10} < 0.8148, & B_7 &\leq \frac{B_{11}}{\varepsilon} < 0.9793, & B_6 &\leq \frac{B_{10}}{\varepsilon} < 1.1589, \\ B_5 &\leq \frac{B_9}{\varepsilon} < 1.4198, & B_4 &\leq \frac{3B_{10}}{2\varepsilon} < 1.7384, & B_3 &\leq \frac{B_{11}}{\varepsilon^2} < 2.0892. \end{aligned} \quad (5.1)$$

Claim(iv) $B_2 > 1.828$, $B_4 > 1.426$, $B_6 > 1.019$ and $B_8 > 0.715$

Suppose $B_2 \leq 1.828$. Then $2(B_2 + B_4 + B_6 + B_8 + B_{10} + B_{12}) < 2(1.828 + 1.7384 + 1.1589 + 0.8148 + 0.5432 + 0.4165) < 13$, giving thereby a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2)_w$.

Similarly we obtain lower bounds on B_4, B_6 and B_8 using $(2, 2, 2, 2, 2, 2)_w$.

Claim(v) $B_2 > 2.0299$

Suppose $B_2 \leq 2.0299$. Consider following two cases:

Case(i) $B_3 > B_4$

We have $B_3 > B_4 > 1.426 > \text{each of } B_5, \dots, B_{12}$. So the inequality $(2, 10^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 10.3(\frac{1}{B_1B_2})^{\frac{1}{10}} > 13$. The left side is a decreasing function of B_1 , so replacing B_1 by B_2 we get $2B_2 + 10.3(\frac{1}{B_2^2})^{\frac{1}{10}} > 13$, which is not true for $B_2 \leq 2.0299$.

Case(ii) $B_3 \leq B_4$

As $B_4 > 1.426 > \text{each of } B_5, \dots, B_{12}$, the inequality $(3, 9^*)$ holds, i.e. $\phi_{13}(X) = 4B_1 - \frac{B_3^3}{X} + 9(\frac{1}{B_1 X})^{\frac{1}{9}} > 13$, where $X = B_2 B_3 < \min\{B_1^2, (2.0299)(1.7384)\} = \alpha$ say. Now $\phi_{13}(X)$ is an increasing function of X for $B_1 \geq B_2 > 1.828$ and so $\phi_{13}(X) < \phi_{13}(\alpha)$, which can be seen to be less than 13. Hence we have $B_2 > 2.0299$.

Claim(vi) $B_1 > 2.17$ and $B_3 < 1.9517$

Using (2.3) we have $B_3 \leq (\frac{\gamma_{10}^{10}}{B_1 B_2})^{\frac{1}{10}} < 1.9648$. Therefore $B_2 > 2.0299 > \text{each of } B_3, \dots, B_{12}$. So the inequality $(1, 11^*)$ holds, i.e. $B_1 + 11.62(\frac{1}{B_1})^{\frac{1}{11}} > 13$. But this is not true for $B_1 \leq 2.17$. So we must have $B_1 > 2.17$. Again using (2.3) we have $B_3 < (\frac{2.2636302^{10}}{2.17 \times 2.0299})^{\frac{1}{10}} < 1.9517$.

Claim(vii) $B_4 < 1.646$

Suppose $B_4 \geq 1.646$. From (5.1) and Claims (i)-(iii), we have $\frac{B_4^3}{B_5 B_6 B_7} > 2$ and $\frac{B_8^3}{B_9 B_{10} B_{11}} > \frac{(\varepsilon B_4)^3}{B_9 B_{10} B_{11}} > 2$. Applying AM-GM to the inequality $(1, 2, 4, 4, 1)$ we get $\phi_{14} = B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 + 4B_8 + B_{12} - \sqrt{B_4^5 B_8^5 B_1 B_2 B_3 B_{12}} > 13$. We find that left side is a decreasing function of B_2 , B_8 and B_{12} . So we can replace B_2 by 2.0299, B_8 by εB_4 and B_{12} by $\varepsilon^2 B_4$. Then it turns a decreasing function of B_4 , so can replace B_4 by 1.646 to find that $\phi_{14} < 13$ for $B_1 < 2.52178703$ and $B_3 < 1.9517$, a contradiction. Hence we have $B_4 < 1.646$.

Claim(viii) $B_1 < 2.4273$

Suppose $B_1 \geq 2.4273$. Consider following two cases:

Case(i) $B_5 > B_6$

Here $B_5 > \text{each of } B_6, \dots, B_{12}$ as $B_5 \geq \varepsilon B_1 > 1.137 > \text{each of } B_7, \dots, B_{12}$. Also $B_2 B_3 B_4 < 2.2254706 \times 1.9517 \times 1.646 < 7.15$. So $\frac{B_1^3}{B_2 B_3 B_4} > 2$. Hence the inequality $(4, 8^*)$ holds. This gives $4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 8(B_1 B_2 B_3 B_4)^{-1/8} > 13$. Left side is an increasing function of $B_2 B_3 B_4$ and decreasing function of B_1 . So we can replace $B_2 B_3 B_4$ by 7.15 and B_1 by 2.4273 to get a contradiction.

Case(ii) $B_5 \leq B_6$

Using (5.1) we have $B_5 \leq B_6 < 1.1589$ and so $B_4 \leq \frac{4}{3} B_5 < 1.5452$. Therefore $\frac{B_2^3}{B_3 B_4 B_5} > 2$ as $B_2 > 2.0299$ and $B_3 < 1.9517$. Also from Claim(iv), $B_6 > 1.019 > \text{each of } B_7, \dots, B_{12}$. Hence the inequality $(1, 4, 7^*)$ holds. This gives $B_1 + 4B_2 - \frac{1}{2} \frac{B_2^4}{B_3 B_4 B_5} + 7(B_1 B_2 B_3 B_4 B_5)^{-1/7} > 13$. Left side is an increasing function of $B_3 B_4 B_5$ and B_1 and a decreasing function of B_2 . One can check that inequality is not true for $B_3 B_4 B_5 < 1.9517 \times 1.5452 \times 1.1589$, $B_1 < 2.5217871$ and for $B_2 > 2.0299$.

Hence we must have $B_1 < 2.4273$.

Claim(ix) $B_5 < 1.396$

Suppose $B_5 \geq 1.396$. From (5.1), $B_6 B_7 B_8 < 0.925$ and $B_{10} B_{11} B_{12} <$

0.104, so we have $\frac{B_5^3}{B_6 B_7 B_8} > 2$ and $\frac{B_9^3}{B_{10} B_{11} B_{12}} > \frac{(\varepsilon B_5)^3}{B_{10} B_{11} B_{12}} > 2$. Applying AM-GM to the inequality (1,2,1,4,4) we get $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + B_4 + 4B_5 + 4B_9 - \sqrt{B_5^5 B_9^5 B_1 B_2 B_3 B_4} > 13$. We find that left side is a decreasing function of B_2 and B_9 . So we replace B_2 by 2.0299 and B_9 by εB_5 . Now it becomes a decreasing function of B_5 and an increasing function of B_1 so replacing B_5 by 1.396 and B_1 by 2.4273, we find that above inequality is not true for $1.522 < B_3 < 1.9517$ and $1.426 < B_4 < 1.646$, giving thereby a contradiction. Hence we must have $B_5 < 1.396$.

Claim(x) $B_3 > 1.7855$

Suppose $B_3 \leq 1.7855$. We have $B_4 > 1.426 > \text{each of } B_5, B_6, \dots, B_{12}$, hence the inequality (1,2,9*) holds. It gives $\phi_{15} = B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 9(\frac{1}{B_1 B_2 B_3})^{\frac{1}{9}} > 13$. It is easy to check that left side of above inequality is a decreasing function of B_2 and an increasing function of B_1 and B_3 . So replacing B_1 by 2.4273, B_3 by 1.7855 and B_2 by 2.0299 we get $\phi_{15} < 13$, a contradiction. Hence we have $B_3 > 1.7855$.

Claim(xi) $B_2 > 2.0733$

Suppose $B_2 \leq 2.0733$. We have $B_3 > 1.7855 > \text{each of } B_4, B_5, \dots, B_{12}$, hence the inequality (2,10*) holds. It gives $\phi_{16} = 4B_1 - \frac{2B_1^2}{B_2} + 10.3(\frac{1}{B_1 B_2})^{\frac{1}{10}} > 13$. The left side is a decreasing function of B_1 and an increasing function of B_2 , so replacing B_1 by 2.17 and B_2 by 2.0733 we get $\phi_{16} < 13$, a contradiction.

Claim(xii) $B_7 < 0.92$ and $B_5 < 1.38$

Suppose $B_7 \geq 0.92$. Here we have $B_4 B_5 B_6 < 2.67$ and $B_8 B_9 B_{10} < 0.295$, so $\frac{B_3^3}{B_4 B_5 B_6} > 2$ and $\frac{B_7^3}{B_8 B_9 B_{10}} > 2$. Also $2B_{11} \geq 2\varepsilon B_7 > B_{12}$. Applying AM-GM to the inequality (2,4,4,2) we get $\phi_{17} = 4B_1 - \frac{2B_1^2}{B_2} + 4B_3 + 4B_7 - \sqrt{B_3^5 B_7^5 B_1 B_2 B_{11} B_{12}} + 4B_{11} - \frac{2B_{11}^2}{B_{12}} > 13$. We find that left side is a decreasing function of B_1 and B_{11} . So we can replace B_1 by 2.17 and B_{11} by εB_7 . Then left side becomes a decreasing function of B_7 and an increasing function of B_2 , so can replace B_7 by 0.92 and B_2 by 2.2254706 to see that $\phi_{17} < 13$ for $1.7855 < B_3 < 1.9517$ and $0.3376 < B_{12} < 0.4156$, a contradiction. Hence $B_7 < 0.92$. Further $B_5 \leq \frac{3}{2}B_7$ gives $B_5 < 1.38$.

Claim(xiii) $B_6 < 1.097$

Suppose $B_6 \geq 1.097$. Here we have $B_3 B_4 B_5 < 4.44$ and $B_7 B_8 B_9 < 0.5$, so $\frac{B_2^3}{B_3 B_4 B_5} > \frac{(2.0733)^3}{4.44} > 2$ and $\frac{B_6^3}{B_7 B_8 B_9} > 2$. Also $2B_{10} \geq 2\varepsilon B_6 > B_{11}$. Applying AM-GM to the inequality (1,4,4,2,1) we get $\phi_{18} = B_1 + 4B_2 + 4B_6 - \sqrt{B_2^5 B_6^5 B_1 B_{10} B_{11} B_{12}} + 4B_{10} - \frac{2B_{10}^2}{B_{11}} + B_{12} > 13$. We find that left side is a decreasing function of B_{10} , B_{12} and B_{11} . So we can replace B_{10} by εB_6 and B_{12} by 0.3376 and B_{11} by $\frac{3\varepsilon}{4}B_6$. Then left side becomes a decreasing function of B_6 , so we can replace B_6 by 1.097 to find that $\phi_{18} < 13$, for

$2.17 < B_1 < 2.4273$ and $2.0733 < B_2 < 2.2254706$, a contradiction. Hence we must have $B_6 < 1.097$.

Claim(xiv) $B_5 > B_6$ and $\frac{B_1^3}{B_2 B_3 B_4} < 2$

First suppose $B_5 \leq B_6$, then $B_4 B_5 B_6 < 1.646 \times 1.097^2 < 1.981$ and $\frac{B_3^3}{B_4 B_5 B_6} > 2$. Also $B_7 \geq \varepsilon B_3 > 0.83 > \text{each of } B_8, \dots, B_{12}$. Hence the inequality $(2, 4, 6^*)$ holds, i.e. $4B_1 - \frac{2B_2^2}{B_3} + 4B_3 - \frac{1}{2} \frac{B_3^4}{B_4 B_5 B_6} + 6(\frac{1}{B_1 B_2 B_3 B_4 B_5 B_6})^{\frac{1}{6}} > 13$. Now the left side is a decreasing function of B_1 and B_3 as well; also it is an increasing function of B_2 and $B_4 B_5 B_6$. But one can check that this inequality is not true for $B_1 > 2.17$, $B_3 > 1.7855$, $B_2 < 2.2254706$ and $B_4 B_5 B_6 < 1.981$, giving thereby a contradiction.

Further suppose $\frac{B_1^3}{B_2 B_3 B_4} \geq 2$, then as $B_5 > B_6 > 1.019 > \text{each of } B_7, \dots, B_{12}$, the inequality $(4, 8^*)$ holds. Now working as in Case(i) of Claim(viii) we get contradiction for $B_1 > 2.17$ and $B_2 B_3 B_4 < 2.2254706 \times 1.9517 \times 1.646 < 7.14934$.

Claim(xv) $B_3 < 1.9$ and $B_1 < 2.4056$

Suppose $B_3 \geq 1.9$, then for $B_4 B_5 B_6 < 1.646 \times 1.38 \times 1.097 < 2.492$, $\frac{B_3^3}{B_4 B_5 B_6} > 2$. Also $B_7 \geq \varepsilon B_3 > 0.89 > \text{each of } B_8, \dots, B_{12}$. Hence the inequality $(2, 4, 6^*)$ holds. Now working as in Claim(xiv) we get contradiction for $B_1 > 2.17$, $B_2 < 2.2254706$, $B_3 > 1.9$ and $B_4 B_5 B_6 < 2.492$. So $B_3 < 1.9$. Further if $B_1 \geq 2.4056$, then $\frac{B_1^3}{B_2 B_3 B_4} > \frac{(2.4056)^3}{2.2254706 \times 1.9 \times 1.646} > 2$, contradicting Claim(xiv).

Claim(xvi) $B_4 < 1.58$ and $B_1 < 2.373$

Suppose $B_4 \geq 1.58$, then for $B_5 B_6 B_7 < 1.38 \times 1.097 \times 0.92 < 1.393$, $\frac{B_4^3}{B_5 B_6 B_7} > 2$. Also $B_8 \geq \varepsilon B_4 > 0.74 > \text{each of } B_9, \dots, B_{12}$. Hence the inequality $(1, 2, 4, 5^*)$ holds, i.e. $\phi_{19} = B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 - \frac{1}{2} \frac{B_4^4}{B_5 B_6 B_7} + 5(\frac{1}{B_1 B_2 B_3 B_4 B_5 B_6 B_7})^{\frac{1}{5}} > 13$. Left side is a decreasing function of B_2 and B_4 . So we replace B_2 by 2.0733 and B_4 by 1.58. Then it becomes an increasing function of B_1 , B_3 and $B_5 B_6 B_7$. So we replace B_1 by 2.4056, B_3 by 1.9 and $B_5 B_6 B_7$ by 1.393 to find that $\phi_{19} < 13$, a contradiction. Further if $B_1 \geq 2.373$, then $\frac{B_1^3}{B_2 B_3 B_4} > 2$, contradicting Claim(xiv).

Final Contradiction:

We have $B_3 B_4 B_5 < 1.9 \times 1.58 \times 1.38 < 4.15$. Therefore $\frac{B_2^3}{B_3 B_4 B_5} > 2$. Also $B_6 > 1.019 > \text{each of } B_7, \dots, B_{12}$. Hence the inequality $(1, 4, 7^*)$ holds. Now we get contradiction working as in Case(ii) of Claim(viii).

5.2 $2.17 < B_{12} < 2.2254706$

Here $B_1 \geq B_{12} > 2.17$. Using (2.1), (2.2), we have

$$B_{11} = (B_1 B_2 \cdots B_{10} B_{12})^{-1} < (\frac{3}{64} \varepsilon^8 B_1^{10} B_{12})^{-1} < 1.8223.$$

Claim(i) Either $B_{11} < 0.4307$ or $B_{11} > 1.818$

Suppose $0.4307 \leq B_{11} \leq 1.818$. The inequality $(10^*, 1, 1)$ gives $10.3(\frac{1}{B_{11}B_{12}})^{\frac{1}{10}} + B_{11} + B_{12} > 13$, which is not true for $0.4307 \leq B_{11} \leq 1.818$ and $2.17 < B_{12} < 2.2254706$. So we must have either $B_{11} < 0.4307$ or $B_{11} > 1.818$.

Claim(ii) $B_{11} < 0.4307$

Suppose $B_{11} \geq 0.4307$, then using Claim(i) we have $B_{11} > 1.818$. Now we have using (2.1),(2.2),

$$B_2 = (B_1 B_3 \cdots B_{12})^{-1} < (\frac{1}{16}\varepsilon^6 B_2^8 B_1 B_{11} B_{12})^{-1}. \text{ This gives } B_2 < 1.777.$$

$$B_3 = (B_1 B_2 B_4 \cdots B_{12})^{-1} < (\frac{3}{64}\varepsilon^4 B_3^7 B_1^2 B_{11} B_{12})^{-1}. \text{ This gives } B_3 < 1.487$$

$$B_4 = (B_1 B_2 B_3 B_5 \cdots B_{12})^{-1} < (\frac{1}{16}\varepsilon^3 B_4^6 B_1^3 B_{11} B_{12})^{-1}. \text{ This gives } B_4 < 1.213.$$

$$B_6 = (B_1 \cdots B_5 B_7 \cdots B_{12})^{-1} < (\frac{1}{16}\varepsilon^2 B_6^4 B_1^5 B_{11} B_{12})^{-1}. \text{ This gives } B_6 < 0.826.$$

$$B_7 = (B_1 \cdots B_6 B_8 \cdots B_{12})^{-1} < (\frac{3}{64}\varepsilon^2 B_7^3 B_1^6 B_{11} B_{12})^{-1}. \text{ This gives } B_7 < 0.697.$$

$$B_8 = (B_1 \cdots B_7 B_9 \cdots B_{12})^{-1} < (\frac{1}{16}\varepsilon^3 B_8^2 B_1^7 B_{11} B_{12})^{-1}. \text{ This gives } B_8 < 0.559.$$

$$B_9 = (B_1 \cdots B_8 B_{10} B_{11} B_{12})^{-1} < (\frac{3}{64}\varepsilon^4 B_9 B_1^8 B_{11} B_{12})^{-1}. \text{ This gives } B_9 < 0.478.$$

$$B_{10} = (B_1 \cdots B_9 B_{11} B_{12})^{-1} < (\frac{1}{16}\varepsilon^6 B_1^9 B_{11} B_{12})^{-1} < 0.359.$$

Therefore we have $\frac{B_1^3}{B_2 B_3 B_4} > 2$ and $B_5 \geq \varepsilon B_1 > 1.01 > \text{each of } B_6, \dots, B_{10}$.

So the inequality $(4, 6^*, 1, 1)$ holds, i.e. $4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 6(B_1 B_2 B_3 B_4 B_{11} B_{12})^{-1/6} + B_{11} + B_{12} > 13$. Now the left side is an increasing function of $B_2 B_3 B_4$, B_{11} and of B_{12} as well. Also it is a decreasing function of B_1 . So we replace $B_2 B_3 B_4$ by $1.777 \times 1.487 \times 1.213$, B_{11} by 1.8223 , B_{12} by 2.2254706 and B_1 by 2.17 to arrive at a contradiction. Hence we must have $B_{11} < 0.4307$.

Claim(iii) $B_{10} < 0.445$

Suppose $B_{10} \geq 0.445$, then $2B_{10} > B_{11}$. So the inequality $(9^*, 2, 1)$ holds, i.e. $\phi_{20} = 9(\frac{1}{B_{10} B_{11} B_{12}})^{\frac{1}{9}} + 4B_{10} - \frac{2B_{10}^2}{B_{11}} + B_{12} > 13$. Now for $B_{10} \geq 0.445$, $B_{11} \geq \frac{3}{4}B_{10}$ and $B_{12} > 2.2254706$, the left side is an increasing function of B_{12} and a decreasing function of B_{10} , so replacing B_{12} by 2.2254706 and B_{10} by 0.445 we find that $\phi_{20} < 13$, for $\frac{3}{4}(0.445) < B_{11} < 0.4307$, a contradiction. Hence we must have $B_{10} < 0.445$.

Using (2.1),(2.2) we have:

$$\begin{aligned} B_9 &\leq \frac{4}{3}B_{10} < 0.594, & B_8 &\leq \frac{3}{2}B_{10} < 0.67, & B_7 &\leq 2B_{10} < 0.89, \\ B_6 &\leq \frac{B_{10}}{\varepsilon} < 0.9494, & B_5 &\leq \frac{4}{3}\frac{B_{10}}{\varepsilon} < 1.266, & B_4 &\leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 1.4242, \\ B_3 &\leq \frac{2B_{10}}{\varepsilon} < 1.899, & B_2 &\leq \frac{B_{10}}{(\varepsilon)^2} < 2.0255. \end{aligned} \quad (5.2)$$

Claim(iv) $B_3 < 1.62$

Suppose $B_3 \geq 1.62$. From (5.2), we have $B_4 B_5 B_6 < 1.712$ and $B_8 B_9 B_{10} < 0.178$, so $\frac{B_3^3}{B_4 B_5 B_6} > 2$ and $\frac{B_7^3}{B_8 B_9 B_{10}} \geq \frac{(\varepsilon B_3)^3}{B_8 B_9 B_{10}} > 2$. Applying AM-GM to the inequality $(2, 4, 4, 1, 1)$ we get $\phi_{21} = 4B_1 - \frac{2B_1^2}{B_2} + 4B_3 + 4B_7 - \sqrt{B_3^5 B_7^5 B_1 B_2 B_{11} B_{12}} +$

$B_{11} + B_{12} > 13$. We find that left side is a decreasing function of B_1 , B_7 and B_{11} . So we can replace B_1 by B_{12} , B_7 by εB_3 and B_{11} by $\varepsilon^2 B_3$. Then it becomes a decreasing function of B_3 , so replacing B_3 by 1.62 we find that $\phi_{21} < 13$, for $1.6275 < B_2 < 2.0255$ and $2.17 < B_{12} < 2.2254706$, a contradiction. Hence we must have $B_3 < 1.62$.

Claim(v) $B_{12} > 2.196$

Suppose $B_{12} \leq 2.196$. From (5.2), we have $B_2 B_3 B_4 < 4.674$ and $\frac{B_1^3}{B_2 B_3 B_4} > 2$. Also $B_5 \geq \varepsilon B_1 > 1.01 > \text{each of } B_6, \dots, B_{11}$. Therefore the inequality $(4, 7^*, 1)$ holds, i.e. $\phi_{22} = 4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 7(B_1 B_2 B_3 B_4 B_{12})^{-1/7} + B_{12} > 13$. Left side is an increasing function of $B_2 B_3 B_4$ and of B_{12} as well. Also it is a decreasing function of B_1 . So we can replace $B_2 B_3 B_4$ by 4.674, B_{12} by 2.196 and B_1 by 2.17 to get $\phi_{22} < 13$, a contradiction. Hence we must have $B_{12} > 2.196$.

Final Contradiction:

Now we have $B_1 \geq B_{12} > 2.196$. We proceed as in Claim(v) and use $(4, 7^*, 1)$. Here we replace $B_2 B_3 B_4$ by 4.674, B_{12} by 2.2254706 and B_1 by 2.196 to get $\phi_{22} < 13$, a contradiction. \square

6 Proof of Theorem 1 for $n = 13$

Here we have $\omega_{13} = 14.455765$, $B_1 \leq \gamma_{13} < 2.6492947$. Using (2.5), we have $l_{13} = 0.3113 < B_{13} < 2.348593 = m_{13}$ and using (2.3) we have $B_2 \leq \gamma_{12}^{12} < 2.348593$.

Claim(i) $B_{13} < 0.3878$

Suppose $B_{13} \geq 0.3878$. The inequality $(12^*, 1)$ gives $13(B_{13})^{-1/12} + B_{13} > 14.455765$. But this inequality is not true for $0.3878 \leq B_{13} < 2.348593$. So we must have $B_{13} < 0.3878$.

Claim(ii) $B_{12} < 0.4353$ and $B_{11} < 0.5804$

Suppose $B_{12} \geq 0.4353$, then $B_{13} \geq \frac{3}{4} B_{12} > 0.3264$ and $2B_{12} > B_{13}$, so $(11^*, 2)$ holds, i.e. $11.62(\frac{1}{B_{12} B_{13}})^{1/11} + 4B_{12} - \frac{2B_{12}^2}{B_{13}} > 14.455765$, which is not true for $B_{12} \geq 0.4353$ and $0.3264 < B_{13} < 0.3878$. Hence we have $B_{12} < 0.4353$. Further $B_{11} \leq \frac{4}{3} B_{12} < \frac{4}{3}(0.4353) < 0.5804$.

Claim(iii) $B_{10} < 0.5942$; $B_8 < 0.8913$; $B_6 < 1.2677$ and $B_4 < 1.9016$

Suppose $B_{10} \geq 0.5942$, then $\frac{B_{10}^3}{B_{11} B_{12} B_{13}} > 2$. So the inequality $(9^*, 4)$ holds. This gives $\psi_1(x) = 9x^{1/9} + 4B_{10} - \frac{1}{2} B_{10}^5 x > 14.455765$, where $x = B_1 B_2 \dots B_9$. The function $\psi_1(x)$ has its maximum value at $x = (\frac{2}{B_{10}^5})^{9/8}$, so $\psi_1(x) < \psi_1((\frac{2}{B_{10}^5})^{9/8}) < 14.455765$ for $0.5942 \leq B_{10} \leq \frac{3}{2} B_{12} < \frac{3}{2}(0.4353) < 0.653$. This gives a contradiction. Hence we have $B_{10} < 0.5942$.

Further $B_8 \leq \frac{3}{2}B_{10} < 0.8913$, $B_6 \leq \frac{B_{10}}{\varepsilon} < 1.2677$ and $B_4 \leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 1.9016$.

Claim(iv) $B_9 < 0.74$ and $B_5 < 1.5788$

Suppose $B_9 \geq 0.74$, then $\frac{B_9^3}{B_{10}B_{11}B_{12}} > 2$. So the inequality $(8^*, 4, 1)$ holds, i.e. $\psi_2(x) = 8x^{1/8} + 4B_9 - \frac{1}{2}B_9^5B_{13}x + B_{13} > 14.455765$, where $x = B_1B_2 \dots B_8$. The function $\psi_2(x)$ has its maximum value at $x = (\frac{2}{B_9^5B_{13}})^{\frac{8}{7}}$, so $\psi_2(x) < \psi_2((\frac{2}{B_9^5B_{13}})^{\frac{8}{7}}) < 14.455765$ for $0.74 \leq B_9 \leq \frac{4}{3}B_{10} < \frac{4}{3}(0.5942) < 0.793$ and $\varepsilon B_9 \leq B_{13} < 0.3878$. This gives a contradiction. Hence $B_9 < 0.74$. and $B_5 \leq \frac{B_9}{\varepsilon} < 1.5788$.

Claim(v) $B_7 < 1.088$

Suppose $B_7 \geq 1.088$, then $\frac{B_7^3}{B_8B_9B_{10}} > 2$. Also $B_{11} \geq \varepsilon B_7 > 0.5$ and $B_{12}B_{13} < 0.4353 \times 0.3878 < 0.169$, we find $B_{11}^2 > B_{12}B_{13}$. So the inequality $(6^*, 4, 3)$ holds. This gives $\psi_3(x) = 6x^{1/6} + 4B_7 - \frac{1}{2}B_7^5B_{11}B_{12}B_{13}x + 4B_{11} - \frac{B_{11}^3}{B_{12}B_{13}} > 14.455765$, where $x = B_1B_2 \dots B_6$. The function $\psi_3(x)$ has its maximum value at $x = (\frac{2}{B_7^5B_{11}B_{12}B_{13}})^{\frac{6}{5}}$, so $\psi_3(x) < \psi_3((\frac{2}{B_7^5B_{11}B_{12}B_{13}})^{\frac{6}{5}}) = 4B_7 + 5(\frac{2}{B_7^5B_{11}B_{12}B_{13}})^{\frac{1}{5}} + 4B_{11} - \frac{B_{11}^3}{B_{12}B_{13}} = \chi(B_{11})$, say. Now for $B_{11} \geq \varepsilon B_7 > 0.5$ and $B_{12}B_{13} < 0.169$, we find that $\chi'(B_{11}) < 0$, so $\chi(B_{11}) \leq \chi(\varepsilon B_7) < 14.455765$ for $1.088 \leq B_7 \leq \frac{3}{2}B_9 < \frac{3}{2}(0.74) < 1.11$ and $\frac{1}{2}(\varepsilon B_7)^2 \leq B_{12}B_{13} < 0.4353 \times 0.3878$. This gives a contradiction.

Claim(vi) $B_2 > 1.942$, $B_4 > 1.538$, $B_6 > 1.103$

Suppose $B_2 \leq 1.942$. Then $2(B_2 + B_4 + B_6 + B_8 + B_{10} + B_{12}) + B_{13} < 2(1.942 + 1.9016 + 1.2677 + 0.8913 + 0.5942 + 0.4353) + 0.3878 < 14.455765$, giving thereby a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2, 1)_w$. Similarly we obtain lower bounds on B_4 and B_6 using $(2, 2, 2, 2, 2, 2, 1)_w$.

Claim(vii) $B_2 > 2.12$

Suppose $B_2 \leq 2.12$. Now we can take $B_4 \geq 1.72$, for if $B_4 < 1.72$, then $2(B_2 + B_4 + B_6 + B_8 + B_{10} + B_{12}) + B_{13} < 2(2.12 + 1.72 + 1.2677 + 0.8913 + 0.5942 + 0.4353) + 0.3878 < 14.455765$, giving thereby a contradiction to $(2, 2, 2, 2, 2, 2, 1)_w$. So we have $B_4 \geq 1.72 > \text{each of } B_5, \dots, B_{13}$. Consider following cases :

Case(i) $B_3 > B_4$

Here $B_3 > B_4 > \text{each of } B_5, \dots, B_{13}$. So the inequality $(2, 11^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 11.62(\frac{1}{B_1B_2})^{\frac{1}{11}} > 14.455765$. The left side is a decreasing function of B_1 , so replacing B_1 by B_2 we get $2B_2 + 11.62(\frac{1}{B_2^2})^{\frac{1}{11}} > 14.455765$, which is not true for $1.942 < B_2 \leq 2.185$.

Case(ii) $B_3 \leq B_4$

Here $B_3 \leq B_4 < 1.9016$. As $B_4 > \text{each of } B_5, \dots, B_{13}$, the inequality $(3, 10^*)$ holds, i.e. $\psi_4(X) = 4B_1 - \frac{B_1^3}{X} + 10.3(\frac{1}{B_1X})^{\frac{1}{10}} > 14.455765$, where $X = B_2B_3 < \alpha = \min\{B_1^2, (2.12)(1.9016)\}$. Now $\psi_4(X)$ is an increasing function

of X for $1.942 < B_1 < 2.6492947$ and $X < \alpha$. Therefore $\psi_4(X) < \psi_4(\alpha)$, which can be seen to be less than 14.455765, a contradiction.

Hence we have $B_2 > 2.12$.

Claim(viii) $B_1 > 2.348593$

Using (2.3), we have $B_3 < (\frac{\gamma_{11}^{11}}{B_1 B_2})^{\frac{1}{11}} < 2.088$. So $B_2 > 2.12 > \text{each of } B_3, \dots, B_{13}$, which implies that the inequality $(1, 12^*)$ holds, i.e. $B_1 + 13(\frac{1}{B_1})^{\frac{1}{12}} > 14.455765$. But this is not true for $B_1 \leq 2.348593$. So we must have $B_1 > 2.348593$.

Claim(ix) $B_3 > 1.865$

Suppose $B_3 \leq 1.865$. Consider following cases :

Case(i) $B_4 > B_5$

As $B_4 > 1.538 > \text{each of } B_6, \dots, B_{13}$, the inequality $(1, 2, 10^*)$ holds, i.e. $\psi_5 = B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 10.3(\frac{1}{B_1 B_2 B_3})^{\frac{1}{10}} > 14.455765$. It is easy to check that ψ_5 is a decreasing function of B_2 and an increasing of B_1 . So we replace B_2 by 2.12 and B_1 by 2.6492947 and find that $\psi_5 < 14.455765$ for $B_3 \leq 1.865$, a contradiction.

Case(ii) $B_4 \leq B_5$

We have $B_4 \leq B_5 < 1.5788$ and $B_3 < 1.865$. So $\frac{B_2^3}{B_3 B_4 B_5} > 2$. Also $B_6 > 1.103 > \text{each of } B_7, \dots, B_{13}$. So the inequality $(1, 4, 8^*)$ holds, i.e. $\psi_6 = B_1 + 4B_2 - \frac{1}{2} \frac{B_2^4}{B_3 B_4 B_5} + 8(B_1 B_2 B_3 B_4 B_5)^{-1/8} > 14.455765$. ψ_6 is an increasing function of $B_3 B_4 B_5$ and B_1 as well and a decreasing function of B_2 . A simple calculation gives $\psi_6 < 14.455765$ for $B_3 B_4 B_5 < 1.865 \times 1.5788^2$, $B_1 < 2.6492947$ and for $B_2 > 2.12$, a contradiction.

Hence we have $B_3 > 1.865$.

Using (2.3) we find $B_4 < (\frac{\gamma_{10}^{10}}{B_1 B_2 B_3})^{\frac{1}{10}} < (\frac{2.2636302^{10}}{(2.348593)(2.12)(1.865)})^{\frac{1}{10}} < 1.812$.

Claim(x) $B_2 > 2.2366$

Suppose $B_2 \leq 2.2366$. As $B_3 > 1.865 > \text{each of } B_4, \dots, B_{13}$, the inequality $(2, 11^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 11.62(\frac{1}{B_1 B_2})^{\frac{1}{11}} > 14.455765$, which is not true for $B_1 \geq B_2$ and $2.12 < B_2 \leq 2.2366$.

Claim(xi) $B_3 > 1.917$ and $B_5 > 1.278$

Suppose $B_3 \leq 1.917$. We work as in Claim(ix) and get a contradiction.

Hence we have $B_3 > 1.917$ and $B_5 \geq \frac{2}{3} B_3 > 1.278$.

Also using (2.3), we find $B_4 < (\frac{(2.2636302)^{10}}{(2.348593)(2.2366)(1.917)})^{\frac{1}{10}} < 1.7969$.

Claim(xii) $B_1 < 2.57$

Suppose $B_1 \geq 2.57$. Using (2.3) we get $B_2 < (\frac{\gamma_{12}^{12}}{B_1})^{\frac{1}{12}} < 2.3311$, $B_3 < (\frac{\gamma_{11}^{11}}{B_1 B_2})^{\frac{1}{11}} < 2.042$ and $B_4 < (\frac{\gamma_{10}^{10}}{B_1 B_2 B_3})^{\frac{1}{10}} < 1.7808$. So $\frac{B_1^3}{B_2 B_3 B_4} > 2$. Also $B_5 > 1.278 > \text{each of } B_6, \dots, B_{13}$. So the inequality $(4, 9^*)$ holds, i.e. $4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 9(B_1 B_2 B_3 B_4)^{-1/9} > 14.455765$. Left side is an increasing function

of $B_2B_3B_4$ and a decreasing function of B_1 . One can check that inequality is not true for $B_2B_3B_4 < 2.3311 \times 2.042 \times 1.7808$ and for $B_1 \geq 2.57$. Hence we have $B_1 < 2.57$.

Claim(xiii) $B_5 < 1.522$

Suppose $B_5 \geq 1.522$. As $\frac{B_5^3}{B_6B_7B_8} > 2$ and $\frac{B_9^3}{B_{10}B_{11}B_{12}} \geq \frac{\varepsilon B_5^3}{B_{10}B_{11}B_{12}} > 2$, the inequality (2,2,4,4,1) holds. This gives $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{2B_3^2}{B_4} + 4B_5 - \frac{1}{2} \frac{B_5^4}{B_6B_7B_8} + 4B_9 - \frac{1}{2} \frac{B_9^4}{B_{10}B_{11}B_{12}} + B_{13} > 14.455765$. Applying AM-GM inequality we get $\psi_7 = 4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{2B_3^2}{B_4} + 4B_5 + 4B_9 + B_{13} - \sqrt{B_5^5 B_9^5 B_1 B_2 B_3 B_4 B_{13}} > 14.455765$. As $B_1 > 2.348593$, $B_2 > 2.2366$, $B_3 > 1.917$, $1.538 < B_4 < 1.7969$, $B_5 \geq 1.522$, $B_9 \geq \varepsilon B_5$ and $\varepsilon^2 B_5 \leq B_{13} < 0.3878$, we find that ψ_7 is a decreasing function of B_1 , B_3 , B_9 and B_{13} . So we can replace B_1 by 2.348593, B_3 by 1.917, B_9 by εB_5 and B_{13} by $\varepsilon^2 B_5$. Now left side becomes a decreasing function of B_5 , so replacing B_5 by 1.522 we find that $\psi_7 < 14.455765$ for $2.2366 < B_2 < 2.348593$ and $1.538 < B_4 < 1.7969$. Hence $B_5 < 1.522$.

Claim(xiv) $B_2 < 2.278$

Suppose $B_2 \geq 2.278$. Using (2.3) we get $B_3 < (\frac{\gamma_{11}^{11}}{B_1 B_2})^{\frac{1}{11}} < 2.055$. Also $B_5 < 1.522$. So $\frac{B_2^3}{B_3 B_4 B_5} > 2$. Also $B_6 > 1.103 > \text{each of } B_7, \dots, B_{13}$. So the inequality (1,4,8*) holds. Proceeding as in Case(ii) of Claim(ix) we find that the inequality is not true for $B_3 B_4 B_5 < 2.055 \times 1.7969 \times 1.522$, $B_1 < 2.57$ and for $B_2 \geq 2.278$. Hence we have $B_2 < 2.278$.

Claim(xv) $B_3 < 1.96$

Suppose $B_3 \geq 1.96$, then $\frac{B_3^3}{B_4 B_5 B_6} > 2$. Also $B_7 \geq \varepsilon B_3 > 0.9187 > \text{each of } B_8, \dots, B_{13}$. So the inequality (2,4,7*) holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{1}{2} \frac{B_3^4}{B_4 B_5 B_6} + 7(B_1 B_2 B_3 B_4 B_5 B_6)^{-1/7} > 14.455765$. Left side is an increasing function of $B_4 B_5 B_6$ and B_2 as well. Also it is decreasing function of B_1 and B_3 as well. One can check that inequality is not true for $B_4 B_5 B_6 < 1.7969 \times 1.522 \times 1.2677$, $B_2 < 2.278$, $B_1 > 2.348593$ and for $B_3 \geq 1.96$. Hence we have $B_3 < 1.96$.

Final Contradiction: As $\frac{B_2^3}{B_3 B_4 B_5} > \frac{(2.2366)^3}{1.96 \times 1.7969 \times 1.522} > 2$, we get contradiction using the inequality (1,4,8*) and working as in Claim(xiv). \square

7 Proof of Theorem 1 for $n = 14$

Here we have $\omega_{14} = 15.955156$, $B_1 \leq \gamma_{14} < 2.7758041$. Using (2.5), we have $l_{14} = 0.2878 < B_{14} < 2.4711931 = m_{14}$ and using (2.3), we have $B_2 \leq \gamma_{13}^{\frac{13}{14}} < 2.4711931$.

Claim(i) $B_{14} < 0.3789$

The inequality $(13^*, 1)$ gives $14.455765(B_{14})^{\frac{-1}{13}} + B_{14} > 15.955156$, which is not true for $0.3789 \leq B_{14} < 2.4711931$. So we must have $B_{14} < 0.3789$.

Claim(ii) $B_{13} < 0.4183$ and $B_{11} < 0.62745$

Suppose $B_{13} \geq 0.4183$, then $B_{14} \geq \frac{3}{4}B_{13} > 0.3137$ and $2B_{13} > B_{14}$, so $(12^*, 2)$ holds, i.e. $13(\frac{1}{B_{13}B_{14}})^{\frac{1}{12}} + 4B_{13} - \frac{2B_{13}^2}{B_{14}} > 15.955156$, which is not true for $B_{13} \geq 0.4183$ and $0.3137 < B_{14} < 0.3789$. Hence we have $B_{13} < 0.4183$ and $B_{11} \leq \frac{3}{2}B_{13} < 0.62745$.

Claim(iii) $B_{12} < 0.4994$

Suppose $B_{12} \geq 0.4994$, then $B_{12}^2 > B_{13}B_{14}$, so $(11^*, 3)$ holds, i.e. $11.62(\frac{1}{B_{12}B_{13}B_{14}})^{\frac{1}{11}} + 4B_{12} - \frac{B_{12}^3}{B_{13}B_{14}} > 15.955156$. Left side is a decreasing function of B_{12} . Replacing B_{12} by 0.4994 we get $11.62(\frac{1}{(0.4994)B_{13}B_{14}})^{\frac{1}{11}} + 4(0.4994) - \frac{(0.4994)^3}{B_{13}B_{14}} > 15.955156$, which is not true for $\frac{3}{4}(0.4994) \leq \frac{3}{4}B_{12} \leq B_{13} < 0.4183$ and $\frac{2}{3}(0.4994) \leq \frac{2}{3}B_{12} \leq B_{14} < 0.3789$. Hence we must have $B_{12} < 0.4994$.

Claim(iv) $B_{10} < 0.669$; $B_8 < 1.0035$; $B_6 < 1.4273$; $B_4 < 2.1409$

Suppose $B_{10} \geq 0.669$, then $\frac{B_{10}^3}{B_{11}B_{12}B_{13}} > 2$. So the inequality $(9^*, 4, 1)$ holds. This gives $\psi_8(x) = 9x^{1/9} + 4B_{10} - \frac{1}{2}B_{10}^5B_{14}x + B_{14} > 15.955156$, where $x = B_1B_2 \dots B_9$. The function $\psi_8(x)$ has its maximum value at $x = (\frac{2}{B_{10}^5B_{14}})^{\frac{9}{8}}$, so $\psi_8(x) < \psi_8((\frac{2}{B_{10}^5B_{14}})^{\frac{9}{8}}) = 4B_{10} + 8(\frac{2}{B_{10}^5B_{14}})^{\frac{1}{8}} + B_{14} < 15.955156$ for $0.669 \leq B_{10} \leq \frac{3}{2}B_{12} < \frac{3}{2}(0.4994) < 0.75$ and $\varepsilon B_{10} \leq B_{14} < 0.3789$. This gives a contradiction.

Further $B_8 \leq \frac{3}{2}B_{10} < 1.0035$, $B_6 \leq \frac{B_{10}}{\varepsilon} < 1.4273$ and $B_4 \leq \frac{3}{2}\frac{B_{10}}{\varepsilon} < 2.1409$.

Claim(v) $B_9 < 0.8233$; $B_7 < 1.23495$; $B_5 < 1.7565$

Suppose $B_9 \geq 0.8233$, then $\frac{B_9^3}{B_{10}B_{11}B_{12}} > 2$. Also $2B_{13} \geq 2(\varepsilon B_9) > 0.77 > B_{14}$. So the inequality $(8^*, 4, 2)$ holds. This gives $\psi_9(x) = 8x^{1/8} + 4B_9 - \frac{1}{2}B_9^5B_{13}B_{14}x + 4B_{13} - \frac{2B_{13}^2}{B_{14}} > 15.955156$, where $x = B_1B_2 \dots B_8$. The function $\psi_9(x)$ has its maximum value at $x = (\frac{2}{B_9^5B_{13}B_{14}})^{\frac{8}{7}}$, so $\psi_9(x) < \psi_9((\frac{2}{B_9^5B_{13}B_{14}})^{\frac{8}{7}}) = 4B_9 + 7(\frac{2}{B_9^5B_{13}B_{14}})^{\frac{1}{7}} + 4B_{13} - \frac{2B_{13}^2}{B_{14}}$, which is a decreasing function of B_{13} , so for $B_{13} \geq \varepsilon B_9$ we have $\psi_9(x) < 4(1 + \varepsilon)B_9 + 7(\frac{2}{\varepsilon B_9^6B_{14}})^{\frac{1}{7}} - \frac{2(\varepsilon B_9)^2}{B_{14}} < 15.955156$ for $0.8233 \leq B_9 \leq \frac{4}{3}B_{10} < \frac{4}{3}(0.669) < 0.892$ and $\frac{3}{4}\varepsilon B_9 \leq B_{14} < 0.3789$. This gives a contradiction.

Further $B_7 \leq \frac{3}{2}B_9 < 1.23495$ and $B_5 \leq \frac{B_9}{\varepsilon} < 1.7565$.

Claim(vi) $B_2 > 1.858$, $B_4 > 1.599$, $B_6 > 1.182$

Suppose $B_2 \leq 1.858$. Then $2(B_2 + B_4 + B_6 + B_8 + B_{10} + B_{12} + B_{14}) < 2(1.858 + 2.1409 + 1.4273 + 1.0035 + 0.669 + 0.4994 + 0.3789) < 15.955156$, giving thereby a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2, 2)_w$.

Similarly we obtain lower bounds on B_4 and B_6 using $(2, 2, 2, 2, 2, 2, 2)_w$.

Claim(vii) $B_2 > 2.24$

Suppose $B_2 \leq 2.24$. Here we can take $B_4 \geq 1.759$, because if $B_4 < 1.759$, then $2(B_2 + B_4 + B_6 + B_8 + B_{10} + B_{12} + B_{14}) < 2(2.24 + 1.759 + 1.4273 + 1.0035 + 0.669 + 0.4994 + 0.3789) < 15.955156$, giving thereby a contradiction to $(2, 2, 2, 2, 2, 2, 2)_w$. So we have $B_4 \geq 1.759 > \text{each of } B_5, \dots, B_{14}$. Also using (2.3) we find $B_4 \leq (\frac{\gamma_{11}^{11}}{B_2 B_3})^{\frac{1}{12}} \leq (\frac{\gamma_{11}^{11}}{(3/4)B_2^2})^{\frac{1}{12}} < (\frac{(2.393347)^{11}}{(3/4)(1.858)^2})^{\frac{1}{12}} < 2.05586$. Consider following cases:

Case(i) $B_3 > B_4$

Here $B_3 > B_4 > \text{each of } B_5, \dots, B_{14}$. So the inequality $(2, 12^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 13(\frac{1}{B_1 B_2})^{\frac{1}{12}} > 15.955156$. The left side is a decreasing function of B_1 , so replacing B_1 by B_2 we get $2B_2 + 13(\frac{1}{B_2^2})^{\frac{1}{12}} > 15.955156$, which is not true for $1.858 < B_2 \leq 2.33$.

Case(ii) $B_3 \leq B_4$

As $B_4 > \text{each of } B_5, \dots, B_{14}$, the inequality $(3, 11^*)$ holds, i.e. $\psi_{10}(X) = 4B_1 - \frac{B_1^3}{X} + 11.62(\frac{1}{B_1 X})^{\frac{1}{11}} > 15.955156$, where $X = B_2 B_3 < \alpha = \min\{B_1^2, (2.24)(2.05586)\}$. Now $\psi'_{10}(X) = \frac{B_1^3}{X^2}(1 - \frac{11.62}{11}(\frac{X^{10}}{B_1^{34}})^{\frac{1}{11}}) > 0$ for $B_1 \geq B_2 > 1.858$ and $X < \alpha$. Therefore we have $\psi_{10}(X) < \psi_{10}(\alpha) < 15.955156$. Hence we have $B_2 > 2.24$.

Claim(viii) $B_1 > 2.471194$

Using (2.3) we find $B_3 < 2.20463$. So $B_2 > 2.24 > \text{each of } B_3, \dots, B_{14}$, which implies that the inequality $(1, 13^*)$ holds, i.e. $B_1 + 14.455765(\frac{1}{B_1})^{\frac{1}{13}} > 15.955156$. But this is not true for $B_1 \leq 2.471194$. Therefore $B_1 > 2.471194$.

Claim(ix) $B_3 > 1.998$

Suppose $B_3 \leq 1.998$. We have $B_4 > 1.599 > \text{each of } B_6, \dots, B_{14}$. Consider following cases:

Case(i) $B_4 > B_5$

Here the inequality $(1, 2, 11^*)$ holds, i.e. $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 11.62(\frac{1}{B_1 B_2 B_3})^{\frac{1}{11}} > 15.955156$. It is easy to check that the left side is a decreasing function of B_2 and an increasing of B_1 and B_3 as well. So we replace B_2 by 2.24, B_1 by 2.7758041 and B_3 by 1.998 and get a contradiction.

Case(ii) $B_4 \leq B_5$ and $B_1 \leq 2.66$

Here $B_4 \leq B_5 < 1.7565$ and $B_5 \geq B_4 > \text{each of } B_6, \dots, B_{14}$. So the inequality $(1, 2, 1, 10^*)$ holds, i.e. $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + B_4 + 10.3(B_1 B_2 B_3 B_4)^{-1/10} > 15.955156$. Left side is an increasing function of B_4 , B_3 and B_1 . Also it is a decreasing function of B_2 . So we replace B_4 by 1.7565, B_3 by 1.998, B_1 by 2.66 and B_2 by 2.24 to get a contradiction.

Case(iii) $B_4 \leq B_5$ and $B_1 > 2.66$

Here $\frac{B_1^3}{B_2 B_3 B_4} > 2$ and $B_5 \geq B_4 > \text{each of } B_6, \dots, B_{14}$. So the inequality $(4, 10^*)$ holds, i.e. $4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 10.3(B_1 B_2 B_3 B_4)^{-1/10} > 15.955156$.

Left side is an increasing function of $B_2B_3B_4$ and a decreasing function of B_1 . But one can check that the inequality is not true for $B_1 > 2.66$ and $B_2B_3B_4 < 2.471194 \times 1.998 \times 1.7565$.

Hence we must have $B_3 > 1.998$.

Using (2.3) we find $B_3 < 2.18665$, $B_4 < 1.92362$ and $B_5 < 1.69842$.

Claim(x) $B_2 < 2.4266$

Suppose $B_2 \geq 2.4266$. We have $B_6 > 1.182 > \text{each of } B_8, \dots, B_{14}$. Consider following cases:

Case(i) $B_6 > B_7$

Here $B_6 > \text{each of } B_7, \dots, B_{14}$ and $\frac{B_2^3}{B_3B_4B_5} > 2$. So the inequality $(1, 4, 9^*)$ holds, i.e. $B_1 + 4B_2 - \frac{1}{2} \frac{B_2^4}{B_3B_4B_5} + 9(B_1B_2B_3B_4B_5)^{-1/9} > 15.955156$. Left side is an increasing function of $B_3B_4B_5$ and B_1 as well and a decreasing function of B_2 . One can check that inequality is not true for $B_3B_4B_5 < 2.18665 \times 1.92362 \times 1.69842$, $B_1 < 2.7758041$ and for $B_2 \geq 2.4266$.

Case(ii) $B_6 \leq B_7$

Here $B_6 \leq B_7 < 1.23495$, $B_5 \leq \frac{4}{3}B_6 < 1.6466$ and $B_4 \leq \frac{3}{2}B_6 < 1.8525$. So $\frac{B_3^3}{B_4B_5B_6} > 2$. Also $B_7 \geq B_6 > \text{each of } B_8, \dots, B_{14}$. So the inequality $(2, 4, 8^*)$ holds, i.e. $\psi_{11} = 4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{1}{2} \frac{B_3^4}{B_4B_5B_6} + 8(B_1B_2B_3B_4B_5B_6)^{-1/8} > 15.955156$. Left side is an increasing function of $B_4B_5B_6$. Also it is decreasing function of B_1 and B_3 as well. We replace $B_4B_5B_6$ by $1.8525 \times 1.6466 \times 1.23495$, B_3 by 1.998 and B_1 by B_2 to find that $\psi_{11} < 15.955156$ for $2.24 < B_2 < 2.4711931$, a contradiction.

Claim(xi) $B_2 > 2.372$ and $B_1 < 2.635$

First suppose $B_2 \leq 2.372$. As $B_3 > 1.998 > \text{each of } B_4, \dots, B_{14}$, the inequality $(2, 12^*)$ holds, i.e. $\psi_{12}(B_1) = 4B_1 - \frac{2B_1^2}{B_2} + 13(\frac{1}{B_1B_2})^{\frac{1}{12}} > 15.955156$. $\psi_{12}(B_1)$ is a decreasing function of B_1 , so for $B_1 > 2.471194$, $\psi_{12}(B_1) < \psi_{12}(2.471194) < 15.955156$ for $2.24 < B_2 \leq 2.372$. So we must have $B_2 > 2.372$.

Further if $B_1 \geq 2.635$, then $\psi_{12}(B_1) \leq \psi_{12}(2.635) < 15.955156$ for $2.24 < B_2 < 2.4266$. So we must have $B_1 < 2.635$.

Claim(xii) $B_3 < 2.097$

Suppose $B_3 \geq 2.097$. We have $B_7 \geq \varepsilon B_3 > 0.9829 > \text{each of } B_9, \dots, B_{14}$. Consider following cases:

Case(i) $B_7 > B_8$

Here $B_7 > \text{each of } B_8, \dots, B_{14}$. Using (2.3) we have $B_4 < 1.90524$ and $B_5 < 1.68058$. Also $B_6 < 1.4273$. So $\frac{B_3^3}{B_4B_5B_6} > 2$, which implies that the inequality $(2, 4, 8^*)$ holds. Now proceeding as in Case(ii) of Claim(x) we get contradiction for $B_4B_5B_6 < 1.90524 \times 1.68058 \times 1.4273$, $B_2 < 2.4266$, $B_1 > 2.471194$ and for $B_3 \geq 2.097$.

Case(ii) $B_7 \leq B_8$

Here $B_7 \leq B_8 < 1.0035$, $B_6 \leq \frac{4}{3}B_7 < 1.338$ and $B_5 \leq \frac{3}{2}B_7 < 1.50525$. So $\frac{B_4^3}{B_5B_6B_7} > 2$. Also $B_8 \geq B_7 >$ each of B_9, \dots, B_{14} . So the inequality $(1, 2, 4, 7^*)$ holds. This gives $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 - \frac{1}{2} \frac{B_4^4}{B_5B_6B_7} + 7(B_1 \dots B_7)^{-1/7} > 15.955156$. It is easy to check that left side of this inequality is a decreasing function of B_2 and B_4 . Also it is increasing function of $B_5B_6B_7$, B_1 and B_3 . But the inequality is not true for $B_2 > 2.372$, $B_4 > 1.599$, $B_1 < 2.635$, $B_3 < 2.18665$ and $B_5B_6B_7 < 1.50525 \times 1.338 \times 1.0035$.

Claim (xiii) $B_4 < B_5$

Suppose $B_4 \geq B_5$. Also $B_4 > 1.599 >$ each of B_6, \dots, B_{14} . So the inequality $(1, 2, 11^*)$ holds, i.e. $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 11.62(\frac{1}{B_1B_2B_3})^{\frac{1}{11}} > 15.955156$. It is easy to check that the left side is a decreasing function of B_2 and an increasing of B_1 and B_3 as well. So we replace B_2 by 2.372, B_1 by 2.635 and B_3 by 2.097 and get a contradiction. Hence we have $B_4 < B_5$.

Final Contradiction:

Using (2.3) we have $B_5 < 1.68872$. Also $B_4 < B_5$, i.e. $1.599 < B_4 < B_5 < 1.68872$. It also gives $B_5 > 1.599 >$ each of B_6, \dots, B_{14} . So the inequality $(2, 2, 10^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{2B_3^2}{B_4} + 10.3(\frac{1}{B_1B_2B_3B_4})^{\frac{1}{10}} > 15.955156$. Now the left side is a decreasing function of B_1 and B_3 . Also it is an increasing function of B_2 and B_4 . But this inequality is not true for $B_1 > 2.471194$, $B_3 > 1.998$, $B_2 < 2.4266$ and $B_4 < 1.68872$. \square

8 Proof of Theorem 1 for $n = 15$

Here we have $\omega_{15} = 17.498499$, $B_1 \leq \gamma_{15} < 2.90147763$. Using (2.5) we have $l_{15} = 0.2667 < B_{15} < 2.5931615 = m_{15}$. Using (2.3) we have $B_2 \leq \gamma_{14}^{\frac{14}{15}} < 2.5931615$ and $B_4^{12} \leq \frac{\gamma_{12}^{12}}{B_1B_2B_3} \leq \frac{\gamma_{12}^{12}}{B_1 \cdot (3/4)B_1 \cdot (2/3)B_1} \leq \frac{2\gamma_{12}^{12}}{B_4^3}$, i.e. $B_4 \leq (2\gamma_{12}^{12})^{\frac{1}{15}} < (2(2.52178703)^{12})^{\frac{1}{15}} < 2.195$.

Claim(i) $B_{15} < 0.3705$

The inequality $(14^*, 1)$ gives $15.955156(B_{15})^{\frac{-1}{14}} + B_{15} > 17.498499$, which is not true for $0.3705 \leq B_{15} < 2.5931615$. So we must have $B_{15} < 0.3705$.

Claim(ii) $B_{14} < 0.4101$

Suppose $B_{14} \geq 0.4101$, then $B_{15} \geq \frac{3}{4}B_{14} > 0.3075$ and $2B_{14} > B_{15}$, so $(13^*, 2)$ holds, i.e. $14.455765(\frac{1}{B_{14}B_{15}})^{\frac{1}{13}} + 4B_{14} - \frac{2B_{14}^2}{B_{15}} > 17.498499$. But this is not true for $B_{14} \geq 0.4101$ and $0.3075 < B_{15} < 0.3705$. Hence $B_{14} < 0.4101$.

Claim(iii) $B_{13} < 0.4813$

Suppose $B_{13} \geq 0.4813$, then $B_{13}^2 > B_{14}B_{15}$, so $(12^*, 3)$ holds, i.e. $13(\frac{1}{B_{13}B_{14}B_{15}})^{\frac{1}{12}} + 4B_{13} - \frac{B_{13}^3}{B_{14}B_{15}} > 17.498499$. Left side is a decreasing function

of B_{13} . Replacing B_{13} by 0.4813 we get $13(\frac{1}{(0.4813)B_{14}B_{15}})^{\frac{1}{12}} + 4(0.4813) - \frac{(0.4813)^3}{B_{14}B_{15}} > 17.498499$, which is not true for $\frac{3}{4}(0.4813) \leq B_{14} < 0.4101$ and $\frac{2}{3}(0.4813) \leq B_{15} < 0.3705$. Hence we must have $B_{13} < 0.4813$.

Claim(iv) $B_{12} < 0.5553$

Suppose $B_{12} \geq 0.5553$, then $\frac{B_{12}^3}{B_{13}B_{14}B_{15}} > 2$. So the inequality $(11^*, 4)$ holds. This gives $\psi_{13}(x) = 11.62(x)^{1/11} + 4B_{12} - \frac{1}{2}B_{12}^5x > 17.498499$, where $x = B_1B_2 \dots B_{11}$. The function $\psi_{13}(x)$ has its maximum value at $x = (\frac{11.62}{11} \times \frac{2}{B_{12}^5})^{\frac{11}{10}}$, so $\psi_{13}(x) < \psi_{13}((\frac{11.62}{11} \times \frac{2}{B_{12}^5})^{\frac{11}{10}}) < 17.498499$ for $0.5553 \leq B_{12} \leq \frac{3}{2}B_{14} < 0.616$.

Claim(v) $B_{11} < 0.6471$; $B_9 < 0.97065$; $B_7 < 1.38054$

Suppose $B_{11} \geq 0.6471$, then $\frac{B_{11}^3}{B_{12}B_{13}B_{14}} > 2$. So the inequality $(10^*, 4, 1)$ holds. This gives $\psi_{14}(x) = 10.3x^{1/10} + 4B_{11} - \frac{1}{2}B_{11}^5B_{15}x + B_{15} > 17.498499$, where $x = B_1B_2 \dots B_{10}$. The function $\psi_{14}(x)$ has its maximum value at $x = (\frac{10.3}{10} \times \frac{2}{B_{11}^5B_{15}})^{\frac{10}{9}}$, so $\psi_{14}(x) < \psi_{14}((\frac{10.3}{10} \times \frac{2}{B_{11}^5B_{15}})^{\frac{10}{9}}) < 17.498499$ for $0.6471 \leq B_{11} \leq \frac{3}{2}B_{13} < 0.72195$ and $\varepsilon B_{11} \leq B_{15} < 0.3705$. This gives a contradiction.

Further $B_9 \leq \frac{3}{2}B_{11} < 0.97065$ and $B_7 \leq \frac{B_{11}}{\varepsilon} < 1.38054$.

Claim(vi) $B_{10} < 0.7525$; $B_8 < 1.12875$; $B_6 < 1.6055$

Suppose $B_{10} \geq 0.7525$, then $\frac{B_{10}^3}{B_{11}B_{12}B_{13}} > \frac{(0.7525)^3}{0.6471 \times 0.5553 \times 0.4813} > 2$. Also $2B_{15} \geq 2(\varepsilon B_{10}) > 0.7 > B_{15}$. So the inequality $(9^*, 4, 2)$ holds. This gives $\psi_{15}(x) = 9x^{1/9} + 4B_{10} - \frac{1}{2}B_{10}^5B_{14}B_{15}x + 4B_{14} - \frac{2B_{14}^2}{B_{15}} > 17.498499$, where $x = B_1B_2 \dots B_9$. The function $\psi_{15}(x)$ has its maximum value at $x = (\frac{2}{B_{10}^5B_{14}B_{15}})^{\frac{9}{8}}$, so $\psi_{15}(x) < \psi_{15}((\frac{2}{B_{10}^5B_{14}B_{15}})^{\frac{9}{8}}) = 4B_{10} + 8(\frac{2}{B_{10}^5B_{14}B_{15}})^{\frac{1}{8}} + 4B_{14} - \frac{2B_{14}^2}{B_{15}}$, which is a decreasing function of B_{14} , so for $B_{14} \geq \varepsilon B_{10}$ we have $\psi_{15}(x) < 4(1 + \varepsilon)B_{10} + 8(\frac{2}{\varepsilon B_{10}^5B_{15}})^{\frac{1}{8}} - \frac{2(\varepsilon B_{10})^2}{B_{15}}$, which is less than 17.498499 for $0.7525 \leq B_{10} < \frac{3}{2}(0.5553) < 0.833$ and $\frac{3}{4}\varepsilon B_{10} \leq B_{15} < 0.3705$. This gives a contradiction. Further $B_8 \leq \frac{3}{2}B_{10} < 1.12875$ and $B_6 \leq \frac{B_{10}}{\varepsilon} < 1.6055$.

Claim(vii) $B_2 > 1.916$, $B_4 > 1.644$, $B_6 > 1.249$ and $B_5 < 1.96235$

Suppose $B_2 \leq 1.916$. Then $2(B_2 + B_4 + B_6 + B_8 + B_{10} + B_{12} + B_{14}) + B_{15} < 2(1.916 + 2.195 + 1.6055 + 1.12875 + 0.7525 + 0.5553 + 0.4101) + 0.3705 < 17.498499$, a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2, 2, 1)_w$.

Similarly we obtain lower bounds on B_4 and B_6 using $(2, 2, 2, 2, 2, 2, 2, 1)_w$. Also using (2.3), we find $B_5 \leq (\frac{\gamma_{11}}{B_2B_3B_4})^{\frac{1}{12}} < (\frac{(2.393347)^{11}}{(3/4)B_2^2B_4})^{\frac{1}{12}} < 1.96235$.

Claim(viii) $B_3 > 1.6$; $B_5 < 1.9449$; $B_4 < 2.15477$

Suppose $B_3 \leq 1.6$, then $B_2 \leq \frac{4}{3}B_3 < 2.134$. Therefore $2B_2 + B_3 + 2(B_5 + B_7 + B_9 + B_{11} + B_{13} + B_{15}) < 2(2.134) + 1.6 + 2(1.96235 + 1.38054 + 0.97065 + 0.6471 + 0.4813 + 0.3705) < 17.498499$, a contradiction to $(2, 1, 2, 2, 2, 2, 2, 2)_w$. Now using (2.3) we find $B_5 < 1.9449$ and $B_4 < 2.1547$.

Claim(ix) $B_2 > 2.384$

Suppose $B_2 \leq 2.384$. Consider following cases:

Case(i) $B_3 > 2.15477$

Here B_3 is larger than each of B_4, \dots, B_{15} . So the inequality (2, 13*) holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 14.455765(\frac{1}{B_1 B_2})^{\frac{1}{13}} > 17.498499$, which is not true for $B_1 \geq B_2$ and $1.916 < B_2 \leq 2.45$.

Case(ii) $B_3 \leq 2.15477$ and $B_4 \geq B_5$

As $B_4 \geq B_5$ and $B_4 > 1.644 > \text{each of } B_6, \dots, B_{15}$, the inequality (3, 12*) holds, i.e. $\psi_{16}(X) = 4B_1 - \frac{B_1^3}{X} + 13(\frac{1}{B_1 X})^{\frac{1}{12}} > 17.498499$, where $X = B_2 B_3 < \alpha = \min\{B_1^2, (2.384)(2.15477)\}$. Now $\psi'_{16}(X) = \frac{B_1^3}{X^2} - \frac{13}{12}(\frac{1}{B_1 X^{13}})^{\frac{1}{12}} = \frac{B_1^3}{X^2}(1 - \frac{13}{12}(\frac{X^{11}}{B_1^{37}})^{\frac{1}{12}}) > 0$ for $B_1 \geq B_2 > 1.916$ and $X < \alpha$. Therefore $\psi_{16}(X) < \psi_{16}(\alpha)$, which can be seen to be less than 17.498499.

Case(iii) $B_3 \leq 2.15477$ and $B_4 < B_5$

Here $B_5 > B_4 > \text{each of } B_6, \dots, B_{15}$. Therefore the inequality (2, 2, 11*) holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{2B_3^2}{B_4} + 11.62(\frac{1}{B_1 B_2 B_3 B_4})^{\frac{1}{11}} > 17.498499$. Left side is a decreasing function of B_1 , so replacing B_1 by B_2 we get $2B_2 + 4B_3 - \frac{2B_3^2}{B_4} + 11.62(\frac{1}{B_2^2 B_3 B_4})^{\frac{1}{11}} > 17.498499$. Now the left side is an increasing function of B_2 , so replacing B_2 by 2.384 we find that the inequality is not true for $1.6 < B_3 \leq 2.15477$ and $1.644 < B_4 < B_5 < 1.9449$.

Hence we must have $B_2 > 2.384$.

Claim(x) $B_1 > 2.5931615$ and $B_5 < 1.8575$

Using (2.3) we find $B_3 < 2.318$. So $B_2 > 2.384 > \text{each of } B_3, \dots, B_{15}$, which implies that the inequality (1, 14*) holds, i.e. $B_1 + 15.955156(\frac{1}{B_1})^{\frac{1}{14}} > 17.498499$. But this is not true for $B_1 \leq 2.5931615$. So we must have $B_1 > 2.5931615$.

Now using (2.3) we find $B_5 < 1.8575$.

Claim(xi) $B_3 > 2.133$

Suppose $B_3 \leq 2.133$. Consider following cases:

Case(I) $B_4 > B_5$

As $B_4 > 1.644 > \text{each of } B_6, \dots, B_{15}$, the inequality (1, 2, 12*) holds, i.e. $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 13(\frac{1}{B_1 B_2 B_3})^{\frac{1}{12}} > 17.498499$. The left side is a decreasing function of B_2 and an increasing of B_1 and B_3 as well. So we replace B_2 by 2.384, B_1 by 2.90147763 and B_3 by 2.133 and get a contradiction.

Case(II) $B_4 \leq B_5$ and $B_1 \leq 2.8$

We have $1.644 < B_4 \leq B_5 < 1.8575$. It implies $B_5 > 1.644 > \text{each of } B_6, \dots, B_{15}$. So the inequality (1, 2, 1, 11*) holds, i.e. $\psi_{17} = B_1 + 4B_2 - \frac{2B_2^2}{B_3} + B_4 + 11.62(B_1 B_2 B_3 B_4)^{-1/11} > 17.498499$. ψ_{17} is an increasing function of B_4 and B_3 . Also it is a decreasing function of B_2 . So we replace B_4 by 1.8575, B_3 by 2.133, B_2 by 2.384 and find that $\psi_{17} < 17.498499$ for

$2.5931615 < B_1 \leq 2.8$.

Case(III) $B_4 \leq B_5$ and $B_1 > 2.8$

Here again $B_4 \leq B_5 < 1.8575$ and $B_5 \geq B_4 > 1.644 > \text{each of } B_6, \dots, B_{14}$. Also $\frac{B_1^3}{B_2 B_3 B_4} > \frac{(2.8)^3}{2.594 \times 2.133 \times 1.8575} > 2$. So the inequality $(4, 11^*)$ holds, i.e. $4B_1 - \frac{1}{2} \frac{B_1^4}{B_2 B_3 B_4} + 11.62(B_1 B_2 B_3 B_4)^{-1/11} > 17.498499$. Left side is an increasing function of $B_2 B_3 B_4$ and a decreasing function of B_1 . But one can check that the inequality is not true for $B_1 > 2.8$ and $B_2 B_3 B_4 < 2.594 \times 2.133 \times 1.8575$. Hence we must have $B_3 > 2.133$.

Using (2.3) we find $B_3 < 2.30289$, $B_4 < 2.03406$ and $B_5 < 1.80946$.

Claim(xii) $B_2 > 2.49$

Suppose $B_2 \leq 2.49$. As $B_3 > 2.133 > \text{each of } B_4, \dots, B_{15}$, the inequality $(2, 13^*)$ holds, i.e. $\psi_{18}(B_1) = 4B_1 - \frac{2B_1^2}{B_2} + 14.455765(\frac{1}{B_1 B_2})^{\frac{1}{13}} > 17.498449$. But $\psi_{18}(B_1) < \psi_{18}(2.5931615) < 17.498499$ for $2.384 < B_2 \leq 2.49$. Hence $B_2 > 2.49$.

Claim(xiii) $B_4 > 1.883$; $B_5 < 1.78611$

Suppose $B_4 \leq 1.883$. We have $B_5 \geq \frac{2}{3}B_3 > \frac{2}{3}(2.133) > 1.422 > \text{each of } B_7, \dots, B_{15}$. Consider following cases:

Case(I) $B_5 > B_6$

Here the inequality $(2, 2, 11^*)$ holds, i.e. $\psi_{19} = 4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{2B_3^2}{B_4} + 11.62(\frac{1}{B_1 B_2 B_3 B_4})^{\frac{1}{11}} > 17.498499$. Now the left side is a decreasing function of B_1 and B_3 , so we replace B_1 by B_2 and B_3 by 2.133 and then find that $\psi_{19} < 17.498499$ for $2.49 < B_2 < 2.5931615$ and $1.644 < B_4 < 1.883$.

Case(II) $B_5 \leq B_6$

Here $B_6 \geq B_5 > \text{each of } B_7, \dots, B_{15}$. Also $B_5 \leq B_6 < 1.6055$. Therefore $\frac{B_3^3}{B_3 B_4 B_5} > 2$. So the inequality $(1, 4, 10^*)$ holds, which gives $B_1 + 4B_2 - \frac{1}{2} \frac{B_2^3}{B_3 B_4 B_5} + 10.3(\frac{1}{B_1 B_2 B_3 B_4 B_5})^{\frac{1}{10}} > 17.498499$. Now the left side is a decreasing function of B_2 and an increasing function of B_1 and $B_3 B_4 B_5$. One can check that inequality is not true for $B_2 > 2.49$, $B_1 < 2.90147763$ and $B_3 B_4 B_5 < 2.30289 \times 1.883 \times 1.6055$. Hence we must have $B_4 > 1.883$.

Further using (2.3) we find $B_5 < 1.781$.

Claim(xiv) $B_2 < 2.5585$

Suppose $B_2 \geq 2.5585$. We have $B_6 > 1.249 > \text{each of } B_8, \dots, B_{15}$. Consider following cases:

Case(I) $B_6 > B_7$

We have $\frac{B_2^3}{B_3 B_4 B_5} > 2$, so the inequality $(1, 4, 10^*)$ holds. Working as in Case(II) of Claim(xiii) we here get contradiction for $B_2 \geq 2.5585$, $B_1 < 2.90147763$ and $B_3 B_4 B_5 < 2.30289 \times 2.03406 \times 1.781$.

Case(II) $B_6 \leq B_7$ and $B_3 \leq 2.2$

As $B_4 > 1.883 > \text{each of } B_5, \dots, B_{15}$, the inequality $(1, 2, 12^*)$ holds. Here working as in Case(I) of Claim(xi) we get contradiction for $B_2 > 2.5585$, $B_1 < 2.90147763$ and $B_3 \leq 2.2$.

Case(III) $B_6 \leq B_7$ and $B_3 > 2.2$

Here $B_6 \leq B_7 < 1.38054$. Also $B_7 \geq B_6 > \text{each of } B_8, \dots, B_{15}$ and $\frac{B_3^3}{B_4 B_5 B_6} > \frac{(2.2)^3}{2.03406 \times 1.781 \times 1.38054} > 2$, so the inequality $(2, 4, 9^*)$ holds. This gives $\psi_{20} = 4B_1 - \frac{2B_1^2}{B_2} + 4B_3 - \frac{1}{2} \frac{B_3^4}{B_4 B_5 B_6} + 9 \left(\frac{1}{B_1 B_2 B_3 B_4 B_5 B_6} \right)^{\frac{1}{9}} > 17.498499$. Here ψ_{20} is a decreasing function of B_3 and B_1 and an increasing function of $B_4 B_5 B_6$. So we replace B_1 by B_2 , B_3 by 2.2 and $B_4 B_5 B_6$ by $2.03406 \times 1.78727 \times 1.38054$ and then find that $\psi_{20} < 17.498499$ for $B_2 < 2.5931615$. Hence we must have $B_2 < 2.5585$.

Claim(xv) $B_1 < 2.797$

Suppose $B_1 \geq 2.797$. As $B_3 > 2.133 > \text{each of } B_4, \dots, B_{15}$, the inequality $(2, 13^*)$ holds. Here working as in Case(I) of Claim(ix) we get contradiction for $B_1 \geq 2.797$ and $B_2 < 2.5585$.

Claim(xvi) $B_3 < 2.2398$

Suppose $B_3 \geq 2.2398$, then $B_7 \geq \varepsilon B_3 > 1.0498 > \text{each of } B_9, \dots, B_{15}$. Using (2.3) we get $B_4 < 2.0185$; $B_5 < 1.7724$ and $B_6 < 1.564$. Now consider following cases:

Case(I) $B_7 > B_8$

Here $B_7 > \text{each of } B_8, \dots, B_{15}$. Also $\frac{B_3^3}{B_4 B_5 B_6} > 2$. Therefore the inequality $(2, 4, 9^*)$ holds. Now working as in Case(III) of Claim(xiv) we get contradiction for $B_1 > B_2$, $B_3 \geq 2.2398$, $B_4 B_5 B_6 < 2.0185 \times 1.7724 \times 1.564$ and $B_2 < 2.5585$.

Case(II) $B_7 \leq B_8$

Here $B_8 \geq B_7 > \text{each of } B_9, \dots, B_{15}$ and $B_7 \leq B_8 < 1.12875$. Also $\frac{B_4^3}{B_5 B_6 B_7} > 2$. So the inequality $(1, 2, 4, 8^*)$ holds. This gives $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 4B_4 - \frac{1}{2} \frac{B_4^4}{B_5 B_6 B_7} + 8 \left(\frac{1}{B_1 B_2 B_3 B_4 B_5 B_6 B_7} \right)^{\frac{1}{8}} > 17.498499$. Now the left side is a decreasing function of B_2 and B_4 . This is also an increasing function of B_1 , B_3 and $B_5 B_6 B_7$. One can check that inequality is not true for $B_2 > 2.49$, $B_4 > 1.883$, $B_1 < 2.797$, $B_3 < 2.30289$ and $B_5 B_6 B_7 < 1.7724 \times 1.564 \times 1.12875$. Hence we must have $B_3 < 2.2398$.

Claim(xvii) $B_1 > 2.705$

Suppose $B_1 \leq 2.705$. As $B_4 > 1.883 > \text{each of } B_5, \dots, B_{15}$, the inequality $(1, 2, 12^*)$ holds. Now working as in Case(I) of Claim(xi) we get contradiction for $B_2 > 2.49$, $B_1 \leq 2.705$ and $B_3 < 2.2398$.

Claim(xviii) $B_2 > 2.525$

Suppose $B_2 \leq 2.525$. As $B_3 > 2.133 > \text{each of } B_4, \dots, B_{15}$, the inequality $(2, 13^*)$ holds. Now working as in Claim(xv) we get contradiction for $B_1 > 2.705$ and $B_2 \leq 2.525$.

Final Contradiction:

Using (2.3) we find $B_4 < 2.0173$ and $B_5 \leq (\frac{\gamma_{11}^{11}}{B_1 B_2 B_3 B_4})^{\frac{1}{11}} < 1.7712$. Also $B_6 > 1.249 > \text{each of } B_8, \dots, B_{15}$. Consider following cases:

Case(I) $B_6 > B_7$

We have $\frac{B_2^3}{B_3 B_4 B_5} > 2$, so the inequality $(1, 4, 10^*)$ holds. Working as in Case(II) of Claim(xiii) we here get contradiction for $B_2 > 2.525$, $B_1 < 2.797$ and $B_3 B_4 B_5 < 2.2398 \times 2.0173 \times 1.7712$.

Case(II) $B_6 \leq B_7$ and $B_3 \leq 2.22$

As $B_4 > 1.883 > \text{each of } B_5, \dots, B_{15}$, the inequality $(1, 2, 12^*)$ holds, i.e. $B_1 + 4B_2 - \frac{2B_2^2}{B_3} + 13(\frac{1}{B_1 B_2 B_3})^{\frac{1}{12}} > 17.498499$. Here working as in Case(I) of Claim(xi) we get contradiction for $B_2 > 2.525$, $B_1 < 2.797$ and $B_3 \leq 2.22$.

Case(III) $B_6 \leq B_7$ and $B_3 > 2.22$

Here $B_7 \geq B_6 > \text{each of } B_8, \dots, B_{15}$ and $B_6 \leq B_7 < 1.38054$. Also $\frac{B_3^3}{B_4 B_5 B_6} > 2$, so the inequality $(2, 4, 9^*)$ holds. Here working as in Case(III) of Claim(xiv) we get contradiction for $B_1 > 2.705$, $B_3 > 2.22$, $B_4 B_5 B_6 < 2.0173 \times 1.7712 \times 1.38054$ and $B_2 < 2.5585$. \square

9 Proof of Theorem 1 for $16 \leq n \leq 33$

In addition of Lemmas 1-7, we shall use the following lemmas also:

Lemma 8. For any integer s , $1 \leq s \leq n-1$

$$B_1 B_2 \cdots B_s \geq \begin{cases} \frac{(0.46873)^{k(2k-2)} B_1^s}{4^k} & \text{if } s = 4k \\ \frac{(0.46873)^{k(2k-1)} B_1^s}{4^k} & \text{if } s = 4k+1 \\ \frac{3(0.46873)^{k(2k)} B_1^s}{4 \times 4^k} & \text{if } s = 4k+2 \\ \frac{(0.46873)^{k(2k+1)} B_1^s}{2 \times 4^k} & \text{if } s = 4k+3. \end{cases}$$

This is Lemma 8 of Hans-Gill et al[14].

Lemma 9. For any integer s , $1 \leq s \leq n-1$

$$B_1 B_2 \cdots B_s \geq \begin{cases} \frac{(0.46873)^{k(2k-2)}}{4^k B_n^{n-s}} & \text{if } n-s = 4k \\ \frac{(0.46873)^{k(2k-1)}}{4^k B_n^{n-s}} & \text{if } n-s = 4k+1 \\ \frac{3(0.46873)^{k(2k)}}{4 \times 4^k B_n^{n-s}} & \text{if } n-s = 4k+2 \\ \frac{(0.46873)^{k(2k+1)}}{2 \times 4^k B_n^{n-s}} & \text{if } n-s = 4k+3. \end{cases}$$

This is Lemma 9 of Hans-Gill et al[14].

If for some s , $1 \leq s \leq n-1$, $B_{s+1} \geq B_{s+j}$ for all j , $2 \leq j \leq n-s$ then the inequality $(s^*, (n-s)^*)$ holds, which gives

$$\phi_{s,n-s}(B_1 B_2 \dots B_s) = \omega_s(B_1 B_2 \dots B_s)^{\frac{1}{s}} + \omega_{n-s} \left(\frac{1}{B_1 B_2 \dots B_s} \right)^{\frac{1}{n-s}} > \omega_n. \quad (9.1)$$

Let $\lambda_s^{(n)}$ be the larger of the lower bounds of $B_1 B_2 \dots B_s$ given in Lemmas 8 and 9 and let $\mu_s^{(n)}$ be the upper bound of $B_1 B_2 \dots B_s$ given in (2.4).

The following two lemmas are respectively the Lemmas 11 & 12 of Hans-Gill et al [14].

Lemma 10 : If for some s , $1 \leq s \leq n-1$, $\phi_{s,n-s}(\lambda_s^{(n)}) \leq \omega_n$ and $\phi_{s,n-s}(\mu_s^{(n)}) \leq \omega_n$, then we must have $B_{s+1} < \max\{B_{s+2}, \dots, B_n\}$.

Proof : Suppose $B_{s+1} \geq \max\{B_{s+2}, \dots, B_n\}$, then the inequality $(s^*, (n-s)^*)$ holds which gives the inequality (9.1). Also the function $\phi_{s,n-s}(x)$ has maximum at one of the end points of the interval in which x lies. For $x = B_1 B_2 \dots B_s$ and $\lambda_s^{(n)} \leq B_1 B_2 \dots B_s \leq \mu_s^{(n)}$, this contradicts the hypothesis.

Remark 2: In all the cases we find that $\max\{\phi_{s,n-s}(\lambda_s^{(n)}), \phi_{s,n-s}(\mu_s^{(n)})\}$ is $\phi_{s,n-s}(\mu_s^{(n)})$.

Lemma 11 : Suppose that for some s , $1 \leq s \leq n-1$, $\phi_{s,n-s}(\lambda_s^{(n)}) \leq \omega_n$, but $\phi_{s,n-s}(\mu_s^{(n)}) > \omega_n$. Let a real number $\sigma_s^{(n)}$ be such that $\lambda_s^{(n)} < \sigma_s^{(n)} < \mu_s^{(n)}$ and $\phi_{s,n-s}(\sigma_s^{(n)}) \leq \omega_n$.

(i) If $B_1 B_2 \dots B_s < \sigma_s^{(n)}$ then $B_{s+1} < \max\{B_{s+2}, \dots, B_n\}$,

(ii) If $B_1 B_2 \dots B_s \geq \sigma_s^{(n)}$ then $B_{s+1} \leq \frac{\mu_{s+1}^{(n)}}{\sigma_s^{(n)}}$.

Proof : In Case (i) Lemma 10 gives the result. In Case (ii), since $B_{s+1} = \frac{B_1 \dots B_{s+1}}{B_1 \dots B_s}$ we use Lemma 6 to get the desired result.

In Sections 9.1-9.18, we give proof of Theorem 1 for $16 \leq n \leq 33$. For the proof, we obtain upper bounds on B_s for different s by applying various inequalities and using Lemmas 10 and 11 and get a final contradiction by applying weak inequality $(2, 2, \dots, 2)$ for even n and $(1, 2, 2, \dots, 2)$ or $(2, 1, 2, \dots, 2)$ or $(2, 2, \dots, 2, 1)$ for odd n .

9.1 $n = 16$

Here we have $\omega_{16} = 19.285$, $B_1 \leq \gamma_{16} < 3.0263937$. Using (2.5), we have $l_{16} = 0.2477 < B_{16} < 2.7145981 = m_{16}$. Using (2.3) we have

$$B_2 \leq \gamma_{15}^{\frac{15}{16}} < 2.7145981, \quad B_4 \leq (2\gamma_{13}^{13})^{\frac{1}{16}} < 2.3047, \quad B_5 \leq (4\gamma_{12}^{12})^{\frac{1}{16}} < 2.1823. \quad (9.2)$$

Claim(i) $B_{16} < 0.2928$

Suppose $B_{16} \geq 0.2928$. The inequality $(15^*, 1)$ gives $17.498499(B_{16})^{\frac{-1}{15}} + B_{16} > 19.285$. But this is not true for $0.2928 \leq B_{16} < 2.7145981$. So we have $B_{16} < 0.2928$.

Using (2.1),(2.2) and Claim(i) we find

$$\max\{B_6, B_7, \dots, B_{16}\} < B_6 \leq \frac{3}{2} \frac{B_{16}}{\varepsilon^2} < 1.9991 \quad (9.3)$$

Claim(ii) $B_3 < \max\{B_4, B_5, \dots, B_{16}\} < 2.3047$

If $B_3 \geq \max\{B_4, B_5, \dots, B_{16}\}$, then the inequality $(2, 14^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + \omega_{14}(\frac{1}{B_1 B_2})^{\frac{1}{14}} > 19.285$. The left side is a decreasing function of B_1 , so replacing B_1 by B_2 we get $2B_2 + (15.955156)(\frac{1}{B_2^2})^{\frac{1}{14}} > 19.285$, which is not true for $\frac{3}{4} < \frac{3}{4}B_1 \leq B_2 < 2.7145981$. Hence $B_3 < \max\{B_4, B_5, \dots, B_{16}\}$, which is < 2.3047 (using (9.2),(9.3)). So we have $B_3 < 2.3047$.

Claim(iii) $B_2 < \max\{B_3, B_4, \dots, B_{16}\} < 2.3047$

As $1 = \lambda_1^{(16)} < B_1 < \mu_1^{(16)} = 3.0263937$, we find that $\max\{\phi_{1,15}(\lambda_1^{(16)}), \phi_{1,15}(\mu_1^{(16)})\} = \phi_{1,15}(\mu_1^{(16)}) < \omega_{16}$, therefore using Lemma 10 we have $B_2 < \max\{B_3, B_4, \dots, B_{16}\}$, which is < 2.3047 (by Claim (ii)).

Claim(iv) $B_4 < \max\{B_5, B_6, \dots, B_{16}\} < 2.1823$; $B_2, B_3 < 2.1823$

As $\frac{1}{2} = \lambda_3^{(16)} < B_1 B_2 B_3 < \mu_3^{(16)} = (3.0264)(2.3047)^2$ and $\max\{\phi_{3,13}(\lambda_3^{(16)}), \phi_{3,13}(\mu_3^{(16)})\} = \phi_{3,13}(\mu_3^{(16)}) < \omega_{16}$, therefore using Lemma 10 we have $B_4 < \max\{B_5, B_6, \dots, B_{16}\}$, which is < 2.1823 (using (9.2),(9.3)). So $\max\{B_5, \dots, B_{16}\} < 2.1823$. Now by Claims (ii) and (iii), we have $B_3 < 2.1823$ and $B_2 < 2.1823$ respectively.

Claim(v) $B_2, B_3, B_4, B_5 < \frac{3}{2} \frac{B_{16}}{\varepsilon^2}$

As $\frac{1}{4} = \lambda_4^{(16)} < B_1 B_2 B_3 B_4 < \mu_4^{(16)} = (3.0264)(2.1823)^3$ and $\max\{\phi_{4,12}(\lambda_4^{(16)}), \phi_{4,12}(\mu_4^{(16)})\} = \phi_{4,12}(\mu_4^{(16)}) < \omega_{16}$, therefore using Lemma 10 we have $B_5 < \max\{B_6, B_7, \dots, B_{16}\} \leq \frac{3}{2} \frac{B_{16}}{\varepsilon^2}$. So $B_5 < \frac{3}{2} \frac{B_{16}}{\varepsilon^2}$. Now again using Claims (ii), (iii) and (iv) we get each of B_5, B_4, B_3 and B_2 is $< \frac{3}{2} \frac{B_{16}}{\varepsilon^2}$.

Final Contradiction:

Now

$$\begin{aligned} & 2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} + 2B_{12} + 2B_{14} + 2B_{16} \\ & \leq 2\left(\frac{3/2}{\varepsilon^2} + \frac{3/2}{\varepsilon^2} + \frac{3/2}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\right)B_{16} < 19.285 \quad \text{for } B_{16} < 0.2928. \end{aligned}$$

This gives a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2, 2, 2)_w$. \square

9.2 $n = 17$

Here we have $\omega_{17} = 21.101$, $B_1 \leq \gamma_{17} < 3.1506793$. Using (2.5) we have $l_{17} = 0.2306 < B_{17} < 2.8355395 = m_{17}$. Using (2.3) we have

$$\begin{aligned} B_2 &\leq \gamma_{16}^{\frac{16}{17}} < 2.8355395, & B_4 &\leq (2\gamma_{14}^{14})^{\frac{1}{17}} < 2.4147, & B_5 &\leq (4\gamma_{13}^{13})^{\frac{1}{17}} < 2.2856 \\ B_6 &\leq \left(\frac{4\gamma_{12}^{12}}{\varepsilon}\right)^{\frac{1}{17}} < 2.1794. \end{aligned} \tag{9.4}$$

Claim(i) $B_{17} < 0.298$

Suppose $B_{17} \geq 0.298$. The inequality $(16^*, 1)$ gives $19.285(B_{17})^{\frac{-1}{16}} + B_{17} > 21.101$. But this is not true for $0.298 \leq B_{17} < 2.8355395$. So we must have $B_{17} < 0.298$.

Claim(ii) $B_{16} < 0.3328$

Suppose $B_{16} \geq 0.3328$. The inequality $(15^*, 2)$ gives $17.498499(B_{16}B_{17})^{\frac{-1}{15}} + 4B_{16} - \frac{2B_{16}^2}{B_{17}} > 21.101$. But this is not true for $0.3328 \leq B_{16} \leq \frac{4}{3}B_{17} < \frac{4}{3}(0.298)$ and $\frac{3}{4}(0.3328) < B_{17} < 0.298$. So we must have $B_{16} < 0.3328$.

Using (2.1),(2.2) and Claims(i), (ii) we find

$$\max\{B_7, B_8, \dots, B_{17}\} < B_7 < \frac{4}{3} \frac{B_{16}}{\varepsilon^2} < 2.0197 \tag{9.5}$$

Claim(iii) $B_3 < \max\{B_4, B_5, \dots, B_{17}\} < 2.4147$

Suppose $B_3 \geq \max\{B_4, B_5, \dots, B_{17}\}$, then the inequality $(2, 15^*)$ gives $4B_1 - \frac{2B_1^2}{B_2} + \omega_{15}(\frac{1}{B_1B_2})^{\frac{1}{15}} > 21.101$. The left side is a decreasing function of B_1 , so replacing B_1 by B_2 we get $2B_2 + \omega_{15}(\frac{1}{B_2^2})^{\frac{1}{15}} > 21.101$, which is not true for $\frac{3}{4} < B_2 < 2.8356$. Hence $B_3 < \max\{B_4, B_5, \dots, B_{17}\}$, which is < 2.4147 (using (9.4),(9.5)). So $B_3 < 2.4147$.

Claim(iv) $B_2 < \max\{B_3, B_4, \dots, B_{17}\} < 2.4147$

As $1 = \lambda_1^{(17)} < B_1 < \mu_1^{(17)} = 3.1507$, we find that $\max\{\phi_{1,16}(\lambda_1^{(17)}), \phi_{1,16}(\mu_1^{(17)})\} = \phi_{1,16}(\mu_1^{(17)}) < \omega_{17}$, therefore using Lemma 10 we have $B_2 < \max\{B_3, B_4, \dots, B_{17}\}$, which is < 2.4147 (using Claim (iii)). So $B_2 < 2.4147$.

Claim(v) $B_4 < \max\{B_5, B_6, \dots, B_{17}\} < 2.2856$; $B_3, B_2 < 2.2856$

Now $\frac{1}{2} = \lambda_3^{(17)} < B_1B_2B_3 < \mu_3^{(17)} = (3.1507)(2.4147)^2$ and $\max\{\phi_{3,14}(\lambda_3^{(17)}), \phi_{3,14}(\mu_3^{(17)})\} = \phi_{3,14}(\mu_3^{(17)}) < \omega_{17}$, therefore using Lemma 10 we have $B_4 < \max\{B_5, B_6, \dots, B_{17}\}$, which is < 2.2856 (using (9.4),(9.5)). So $B_4 < 2.2856$. Now again using Claims (iii), (iv) respectively, we have $B_3 < 2.2856$ and $B_2 < 2.2856$.

Claim(vi) $B_5 < \max\{B_6, B_7, \dots, B_{17}\} < 2.1794$; $B_4, B_3, B_2 < 2.1794$

As $\frac{1}{4} = \lambda_4^{(17)} < B_1 B_2 B_3 B_4 < \mu_4^{(17)} = (3.1507)(2.2856)^3$ and $\max\{\phi_{4,13}(\lambda_4^{(17)}), \phi_{4,13}(\mu_4^{(17)})\} = \phi_{4,13}(\mu_4^{(17)}) < \omega_{17}$, therefore using Lemma 10 we have $B_5 < \max\{B_6, B_7, \dots, B_{17}\}$, which is < 2.1794 (using (9.4),(9.5)). So $B_5 < 2.1794$. Now again using Claims (iii), (iv) and (v), we get each of B_4, B_3 and B_2 is < 2.1794 .

Claim(vii) B_6, B_5, B_4, B_3 and $B_2 < \frac{4}{3} \frac{B_{16}}{\varepsilon^2}$

As $\frac{\varepsilon}{4} = \lambda_5^{(17)} < B_1 B_2 B_3 B_4 B_5 < \mu_5^{(17)} = (3.1507)(2.1794)^4$ and $\max\{\phi_{5,12}(\lambda_5^{(17)}), \phi_{5,12}(\mu_5^{(17)})\} = \phi_{5,12}(\mu_5^{(17)}) < \omega_{17}$, therefore using Lemma 10 we have $B_6 < \max\{B_7, B_8, \dots, B_{17}\} < \frac{4}{3} \frac{B_{16}}{\varepsilon^2}$. Now again using Claims (iii)-(vi), we get each of B_5, B_4, B_3 and B_2 is $< \frac{4}{3} \frac{B_{16}}{\varepsilon^2}$.

Final Contradiction

Now $2B_2 + 2B_4 + 2B_6 + 2B_8 + 2B_{10} + 2B_{12} + 2B_{14} + 2B_{16} + B_{17} < 2(\frac{4/3}{\varepsilon^2} + \frac{4/3}{\varepsilon^2} + \frac{4/3}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{16} + B_{17} < 21.101$ for $B_{16} < 0.3328$ and $B_{17} < 0.298$. This gives a contradiction to the weak inequality $(2, 2, 2, 2, 2, 2, 2, 2, 1)_w$. \square

9.3 $n = 18$

Here we have $\omega_{18} = 22.955$, $B_1 \leq \gamma_{18} < 3.2743307$. Using (2.5), we have $l_{18} = 0.2150 < B_{18} < 2.9560725 = m_{18}$.

Claim(i) $B_{18} < 0.3$

Suppose $B_{18} \geq 0.3$. The inequality $(17^*, 1)$ gives $21.101(B_{18})^{\frac{-1}{17}} + B_{18} > 22.955$. But this is not true for $0.3 \leq B_{18} < 2.9560725$. So we must have $B_{18} < 0.3$.

Claim(ii) $B_{16} < 0.385$

The inequality $(15^*, 3)$ gives $17.498499(B_{16}B_{17}B_{18})^{\frac{-1}{15}} + 4B_{16} - \frac{B_{16}^3}{B_{17}B_{18}} > 22.955$. But this is not true for $0.385 \leq B_{16} \leq \frac{3}{2}B_{18} < 0.45$ and $\frac{1}{2}B_{16}^2 \leq B_{17}B_{18} \leq \frac{4}{3}B_{18}^2 < \frac{4}{3}(0.3)^2$. So we must have $B_{16} < 0.385$.

Claim(iii) $B_9, B_{10}, B_{11} \leq \frac{4}{3} \frac{B_{16}}{\varepsilon}$ and $B_2, B_3, B_4, B_5 < \frac{4}{3} \frac{B_{16}}{(\varepsilon)^2}$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(18)} < B_1 < \mu_1^{(18)} = 3.2743307, \\ \frac{3}{4} &= \lambda_2^{(18)} < B_1 B_2 < \mu_2^{(18)} = (1.1519977)^{16}, \\ \frac{1}{2} &= \lambda_3^{(18)} < B_1 B_2 B_3 < \mu_3^{(18)} = (1.2402616)^{15}, \\ \frac{3\varepsilon^8}{4^3 B_{18}^{10}} &< \lambda_8^{(18)} < B_1 B_2 \cdots B_8 \leq \mu_8^{(18)} < (1.8635658)^{10}, \\ \frac{\varepsilon^6}{4^2 B_{18}^9} &< \lambda_9^{(18)} < B_1 B_2 \cdots B_9 \leq \mu_9^{(18)} < (2.0406455)^9. \end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(18)}), \phi_{s,n-s}(\mu_s^{(18)})\}$, for $s = 1, 2, 3, 8, 9\} = \phi_{1,n-1}(\mu_s^{(18)})$, which is < 22.955 . Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{18}\}, \text{ for } i = 2, 3, 4, 9, 10 \quad (9.6)$$

Using (2.1),(2.2), we have $B_{11} \leq \frac{4}{3} \frac{B_{16}}{\varepsilon}$. So $B_9, B_{10} < \frac{4}{3} \frac{B_{16}}{\varepsilon}$ (by (9.6)). Further $B_5 \leq \frac{B_9}{\varepsilon} \leq \frac{4}{3} \frac{B_{16}}{(\varepsilon)^2}$. So each of B_2, B_3 and B_4 is $< \frac{4}{3} \frac{B_{16}}{(\varepsilon)^2}$.

Claim(iv) $B_2, B_3, B_4, B_5, B_6 < \frac{8}{3} \frac{B_{16}}{\varepsilon}$

As $B_1 B_2 B_3 B_4 < (3.2745)(\frac{4}{3} \frac{B_{16}}{(\varepsilon)^2})^3 < 41.765$, i.e. $0.25 = \lambda_4^{(18)} < B_1 B_2 B_3 B_4 < \mu_4^{(18)} = 41.765$, we find that $\max\{\phi_{4,14}(\lambda_4^{(18)}), \phi_{4,14}(\mu_4^{(18)})\} = \phi_{4,14}(\mu_4^{(18)})$, which is < 22.955 . Therefore $B_5 < \max\{B_6, B_7, \dots, B_{18}\}$. We have $B_6 \leq 2B_9 \leq \frac{8}{3} \frac{B_{16}}{\varepsilon}$. Therefore using (9.6) we get B_2, B_3, B_4 and B_5 is $< \frac{8}{3} \frac{B_{16}}{\varepsilon}$.

Final Contradiction

Now $2B_2 + 2B_4 + 2B_6 + \dots + 2B_{18} < 2(\frac{8/3}{\varepsilon} + \frac{8/3}{\varepsilon} + \frac{8/3}{\varepsilon} + \frac{(4/3)^2}{\varepsilon} + \frac{4/3}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{16} + 2B_{18} < 22.955$ for $B_{16} < 0.385$ and $B_{18} < 0.3$. This gives a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.4 $n = 19$

Here we have $\omega_{19} = 24.691$, $B_1 \leq \gamma_{19} < 3.3974439$. Using (2.5), we have $l_{19} = 0.2009 < B_{19} < 3.0761736 = m_{19}$ and using (2.3) we have

$$\begin{aligned} B_2 &\leq \gamma_{18}^{\frac{18}{19}} < 3.0761736, \quad B_4 \leq (2\gamma_{16}^{16})^{\frac{1}{19}} < 2.635321, \\ B_5 &\leq (4\gamma_{15}^{15})^{\frac{1}{19}} < 2.4940956, \quad B_6 \leq (\frac{4\gamma_{14}^{14}}{\varepsilon})^{\frac{1}{19}} < 2.3752798. \end{aligned} \quad (9.7)$$

Claim(i) $B_{19} < 0.348$

Suppose $B_{19} \geq 0.348$. The inequality $(18^*, 1)$ gives $22.955(B_{19})^{\frac{-1}{18}} + B_{19} > 24.691$. But this is not true for $0.348 \leq B_{19} < 3.0761736$. So we have $B_{19} < 0.348$.

Claim(ii) $B_{17} < 0.403$

Suppose $B_{17} \geq 0.403$. Then $B_{18} B_{19} \leq \frac{4}{3} B_{19}^2 < \frac{4}{3} (0.348)^2 < B_{17}^2$. So the inequality $(16^*, 3)$ holds, i.e. $19.285(B_{17} B_{18} B_{19})^{\frac{-1}{16}} + 4B_{17} - \frac{B_{17}^3}{B_{18} B_{19}} > 24.691$. But this is not true for $0.403 \leq B_{17} \leq \frac{3}{2} B_{19} < 0.522$ and $\frac{1}{2} B_{17}^2 \leq B_{18} B_{19} < \frac{4}{3} (0.348)^2$. So we must have $B_{17} < 0.403$.

Claim(iii) $B_{10} < \max\{B_{11}, B_{12}, \dots, B_{19}\} < 1.2897$

Using (2.4) and Lemma 9 we have

$$\begin{aligned} \frac{\varepsilon^{10}}{2 \times 4^2 B_{19}^{11}} &< \lambda_8^{(19)} \leq B_1 B_2 \dots B_8 < \mu_8^{(19)} = (1.827901)^{11}, \\ \frac{3\varepsilon^8}{4^3 B_{19}^{10}} &< \lambda_9^{(19)} \leq B_1 B_2 \dots B_9 < \mu_9^{(19)} = (1.994589)^{10}. \end{aligned}$$

We find that $\max\{\phi_{9,10}(\lambda_9^{(19)}), \phi_{9,10}(\mu_9^{(19)})\} = \phi_{9,10}(\mu_9^{(19)})$, which is < 24.691 . Therefore using Lemma 10 we have $B_{10} < \max\{B_{11}, B_{12}, \dots, B_{19}\}$. From (2.1),(2.2), we find that $B_{11} \leq \frac{3}{2} \frac{B_{17}}{\varepsilon}$ and each of B_{12}, \dots, B_{17} is less than $\frac{3}{2} \frac{B_{17}}{\varepsilon} < 1.2897$. Also $B_{19} < 0.348$, $B_{18} < \frac{4}{3} B_{19}$. Therefore $\max\{B_{11}, B_{12}, \dots, B_{19}\} < 1.2897$.

Claim(iv) $B_9 < 1.31015$

We see that $\phi_{8,11}(\lambda_8^{(19)}) < \omega_{19}$ but $\phi_{8,11}(\mu_8^{(19)}) > \omega_{19}$. So we apply Lemma 11 with $\sigma_8^{(19)} = (1.8278)^{11}$. Here $\phi_{8,11}(\sigma_8^{(19)}) < \omega_{19}$.

In Case(i), when $B_1 B_2 \cdots B_8 < (1.8278)^{11}$, we have $B_9 < \max\{B_{10}, B_{11}, \dots, B_{19}\} < 1.2897$.

In Case(ii), when $B_1 B_2 \cdots B_8 \geq (1.8278)^{11}$, then $B_9 < \frac{\mu_9^{(19)}}{\sigma_8^{(19)}} < \frac{(1.994589)^{10}}{(1.8278)^{11}} < 1.31015$.

Hence $B_9 < 1.31015$.

Claim(v) $B_3 < \max\{B_4, B_5, \dots, B_{19}\} < 2.635321$

Now suppose $B_3 \geq \max\{B_4, B_5, \dots, B_{19}\}$, then the inequality (2, 17*) gives $4B_1 - \frac{2B_1^2}{B_2} + 21.101(\frac{1}{B_1 B_2})^{\frac{1}{17}} > 24.691$. The left side is decreasing function of B_1 , so replacing B_1 by B_2 we find that the inequality is not true for $\frac{3}{4} \leq B_2 < 3.0761736$. Hence we must have $B_3 < \max\{B_4, B_5, \dots, B_{19}\}$, which is < 2.635321 as $B_8 < \frac{4}{3}B_9$, $B_7 < \frac{3}{2}B_9$, $B_9 < 1.31015$ and $B_{10} < \max\{B_{11}, B_{12}, \dots, B_{19}\} < 1.2897$.

Claim(vi) $B_1 > 1.29$

For if $B_1 \leq 1.29$, then $B_i \leq B_1 \leq 1.29$ for each $i = 2, 3, \dots, 19$ and then the inequality $(1, 1, \dots, 1)$ gives a contradiction.

Claim(vii) $B_4 < \max\{B_5, B_6, \dots, B_{19}\} < 2.4940956$, $B_3 < 2.4940956$

Suppose $B_4 \geq \max\{B_5, B_6, \dots, B_{19}\}$, then the inequality (3, 16*) holds, i.e. $\eta(X) = 4B_1 - \frac{B_1^3}{X} + 19.285(\frac{1}{B_1 X})^{\frac{1}{16}} > 24.691$, where $X = B_2 B_3$. $\eta'(X) = \frac{B_1^3}{X^2}(1 - \frac{19.285}{16}(\frac{X^{15}}{B_1^{49}})^{\frac{1}{16}}) > 0$ for $1.29 < B_1 < 3.3974439$ and $0.5 \leq X \leq \alpha$, where $\alpha = \min\{B_1^2, (3.0761736)(2.635321)\}$. Therefore $\eta(X) < \eta(\alpha)$, which is < 24.691 for $1 < B_1 < 3.3974439$, giving thereby a contradiction. Hence we have $B_4 < \max\{B_5, B_6, \dots, B_{19}\}$, which is < 2.4940956 . So $B_4 < 2.4940956$ and hence $B_3 < 2.4940956$ using Claim(v).

Claim(viii) $B_5 < \max\{B_6, B_7, \dots, B_{19}\} < 2.3752798$; $B_3, B_4 < 2.3752798$

As $0.25 \leq \lambda_4^{(19)} < B_1 B_2 B_3 B_4 < \mu_4^{(19)} = (3.3974439)(3.0761736)(2.4940956)^2$, we find that $\max\{\phi_{4,15}(\lambda_4^{(19)}), \phi_{4,15}(\mu_4^{(19)})\} = \phi_{4,15}(\mu_4^{(19)})$, which is < 24.691 . Therefore using Lemma 10 we have $B_5 < \max\{B_6, B_7, \dots, B_{19}\}$, which is < 2.3752798 . Therefore each of B_3, B_4 and B_5 is < 2.3752798 .

Claim(ix) $B_6 < \max\{B_7, B_8, \dots, B_{19}\} < 1.9654$; $B_3, B_4, B_5 < 1.9654$

As $(0.25)(\varepsilon) = \lambda_5^{(19)} < B_1 B_2 B_3 B_4 B_5 < \mu_5^{(19)} = (3.3974439)(3.0761736)(2.3752798)^3$, we find that $\max\{\phi_{5,14}(\lambda_5^{(19)}), \phi_{5,14}(\mu_5^{(19)})\} = \phi_{5,14}(\mu_5^{(19)})$, which is < 24.691 . Therefore using Lemma 10 we have $B_6 < \max\{B_7, B_8, \dots, B_{19}\}$, which is $\leq \frac{3}{2}B_9 < \frac{3}{2}(1.31015)$. Therefore each of B_6, B_5, B_4 and B_3 is $< \frac{3}{2}(1.31015) < 1.9654$.

Claim(x) $B_7 < \max\{B_8, B_9, \dots, B_{19}\} < 1.747$; B_6, B_5, B_4 and B_3 is

< 1.747

Now $\frac{3}{16}(\varepsilon^2) = \lambda_6^{(19)} < B_1 B_2 B_3 B_4 B_5 B_6 < \mu_6^{(19)} = (3.3974439)(3.0761736)(1.9654)^4$.

We find that $\max\{\phi_{6,13}(\lambda_6^{(19)}), \phi_{6,13}(\mu_6^{(19)})\} = \phi_{6,13}(\mu_6^{(19)})$, which is < 24.691 .

Therefore using Lemma 10 we have $B_7 < \max\{B_8, B_9, \dots, B_{19}\}$, which is $\leq \frac{4}{3}B_9 < \frac{4}{3}(1.31015) < 1.747$. Therefore each of B_7, B_6, B_5, B_4 and B_3 is < 1.747 .

Final Contradiction

Now

$B_1 + 2B_3 + 2B_5 + 2B_7 + \dots + 2B_{17} + 2B_{19} < 3.3974439 + 2(3 \times 1.747) + 2(1.31015) + 2(\frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{17} + 2B_{19} < 24.691$ for $B_{17} < 0.403$ and $B_{19} < 0.348$, giving thereby a contradiction to the weak inequality $(1, 2, \dots, 2, 2)_w$. \square

9.5 $n = 20$

Here we have $\omega_{20} = 26.629$, $B_1 \leq \gamma_{20} < 3.520062$. Using (2.5), we have $l_{20} = 0.1880 < B_{20} < 3.195912 = m_{20}$ and using (2.3) we have $B_2 \leq \gamma_{19}^{\frac{19}{20}} < 3.195912$.

Claim(i) $B_{20} < 0.294$

Suppose $B_{20} \geq 0.294$. The inequality $(19^*, 1)$ gives $24.691(B_{20})^{\frac{-1}{19}} + B_{20} > 26.629$. But this is not true for $0.294 \leq B_{20} < 3.195912$. So we must have $B_{20} < 0.294$.

Claim(ii) $B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{20}\}$, for $i = 2, 6, 7, 8, 9, 10$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(20)} < B_1 < \mu_1^{(20)} = 3.520062, \\ \frac{1}{4}\varepsilon &= \lambda_5^{(20)} < B_1 B_2 B_3 B_4 B_5 < \mu_5^{(20)} = (1.4183645)^{15}, \\ \frac{3}{8}\varepsilon^2 &= \lambda_6^{(20)} < B_1 B_2 B_3 B_4 B_5 B_6 < \mu_6^{(20)} = (1.530496)^{14}, \\ \frac{\varepsilon^{15}}{4^3 B_{20}^{13}} &= \lambda_7^{(20)} < B_1 \dots B_7 < \mu_7^{(20)} = (1.6555371)^{13}, \\ \frac{\varepsilon^{12}}{4^3 B_{20}^{12}} &= \lambda_8^{(20)} < B_1 \dots B_8 < \mu_8^{(20)} = (1.7955594)^{12}, \\ \frac{\varepsilon^{10}}{2 \times 4^2 B_{20}^{11}} &= \lambda_9^{(20)} < B_1 \dots B_9 < \mu_9^{(20)} = (1.9530741)^{11}. \end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(20)}), \phi_{s,n-s}(\mu_s^{(20)})\}$, for $s = 1, 5, 6, 7, 8, 9\} = \phi_{1,19}(\mu_1^{(20)})$, which is $< \omega_{20}$. Therefore using Lemma 10 we have $B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{20}\}$, for $i = 2, 6, 7, 8, 9, 10$.

Claim(iii) $B_3 < \max\{B_4, B_5, \dots, B_{20}\}$

Suppose $B_3 \geq \max\{B_4, B_5, \dots, B_{20}\}$, then the inequality $(2, 18^*)$ holds, i.e. $4B_1 - \frac{2B_1^2}{B_2} + 22.955(\frac{1}{B_1 B_2})^{\frac{1}{18}} > 26.629$. The left side is decreasing function of B_1 , so replacing B_1 by B_2 we find that the inequality is not true for $\frac{3}{4} \leq B_2 < 3.195912$. Hence $B_3 < \max\{B_4, B_5, \dots, B_{20}\}$.

Now using Claim(ii) and (iii),

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{20}\}, \text{ for } i = 2, 3, 6, 7, 8, 9, 10. \quad (9.8)$$

From (2.1), (2.2) and Claim (i) we find that, $\max\{B_{11}, B_{12}, \dots, B_{20}\} < B_{11} \leq \frac{4}{3} \frac{B_{20}}{\varepsilon^2}$. So using (9.8) we get each of $B_{10}, B_9, B_8, B_7, B_6$ is $< \frac{4}{3} \frac{B_{20}}{\varepsilon^2}$ and so $B_5 \leq \frac{4}{3} B_6 < \frac{16}{9} \frac{B_{20}}{\varepsilon^2}$ and $B_4 \leq \frac{3}{2} B_6 < \frac{2B_{20}}{\varepsilon^2}$. Using (9.8) we have $B_3 < \frac{2B_{20}}{\varepsilon^2}$ and hence $B_2 < \frac{2B_{20}}{\varepsilon^2}$.

We have now

$$\begin{aligned} \frac{1}{2} &= \lambda_3^{(20)} < B_1 B_2 B_3 < \mu_3^{(20)} = (3.520062) \left(\frac{2(0.294)}{\varepsilon^2} \right)^2 \text{ and} \\ \frac{1}{4} &= \lambda_4^{(20)} < B_1 B_2 B_3 B_4 < \mu_4^{(20)} = (3.520062) \left(\frac{2(0.294)}{\varepsilon^2} \right)^3. \end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(20)}), \phi_{s,n-s}(\mu_s^{(20)})\}$, for $s = 3, 4\} = \phi_{4,16}(\mu_4^{(20)})$, which is $< \omega_{20}$. Therefore using Lemma 10 we have $B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{20}\}$, for $i = 4, 5$. Using it together with (9.8) we get each of B_2, B_3, \dots, B_{10} is $< \frac{4}{3} \frac{B_{20}}{\varepsilon^2}$.

Final Contradiction

Now $2B_2 + 2B_4 + \dots + 2B_{20} < 2(5 \times \frac{4/3}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{20} < 26.629$ for $B_{20} < 0.294$, giving thereby a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.6 $n = 21$

Here we have $\omega_{21} = 28.605$, $B_1 \leq \gamma_{21} < 3.6422432$. Using (2.5), we have $l_{21} = 0.1762 < B_{21} < 3.3153098 = m_{21}$.

Claim(i) $B_{21} < 0.2938$

Suppose $B_{21} \geq 0.2938$. The inequality $(20^*, 1)$ gives $26.629(B_{21})^{\frac{-1}{21}} + B_{21} > 28.605$. But this is not true for $0.2938 \leq B_{21} \leq 3.3153098$. So we must have $B_{21} < 0.2938$.

Claim(ii) $B_{19} < 0.4$

Suppose $B_{19} \geq 0.4$. The inequality $(18^*, 3)$ gives $22.955(B_{19}B_{20}B_{21})^{\frac{-1}{18}} + 4B_{19} - \frac{B_{19}^3}{B_{20}B_{21}} > 28.605$. But this is not true for $0.4 \leq B_{19} \leq \frac{3}{2}B_{21} < 0.4407$ and $\frac{1}{2}B_{19}^2 \leq B_{20}B_{21} \leq \frac{4}{3}B_{21}^2 < \frac{4}{3}(0.2938)^2$. So we must have $B_{19} < 0.4$.

Claim(iii) $B_2, B_3, \dots, B_{10} < \frac{B_{19}}{\varepsilon^2}$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(21)} < B_1 < \mu_1^{(21)} = 3.6422432, \\ \frac{3}{4} &= \lambda_2^{(21)} < B_1 B_2 < \mu_2^{(21)} = (1.1398163)^{19}, \\ \frac{1}{4}\varepsilon &= \lambda_5^{(21)} < B_1 \cdots B_5 < \mu_5^{(21)} = (1.4053838)^{16}, \\ \frac{3}{16}\varepsilon^2 &= \lambda_6^{(21)} < B_1 \cdots B_6 < \mu_6^{(21)} = (1.5130617)^{15}, \end{aligned}$$

$$\begin{aligned}\frac{3\varepsilon^{18}}{4^4 B_{21}^{14}} &= \lambda_7^{(21)} < B_1 \cdots B_7 < \mu_7^{(21)} = (1.6326792)^{14}, \\ \frac{\varepsilon^{15}}{4^3 B_{21}^{13}} &= \lambda_8^{(21)} < B_1 \cdots B_8 < \mu_8^{(21)} = (1.7660691)^{13}, \\ \frac{\varepsilon^{12}}{4^3 B_{21}^{12}} &= \lambda_9^{(21)} < B_1 \cdots B_9 < \mu_9^{(21)} = (1.915440)^{12}.\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(21)}), \phi_{s,n-s}(\mu_s^{(21)})\}$, for $s = 1, 2, 5, 6, 7, 8, 9\} = \phi_{1,20}(\mu_1^{(21)})$, which is $< \omega_{21}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{21}\}, \text{ for } i = 2, 3, 6, 7, 8, 9, 10. \quad (9.9)$$

From (2.1), (2.2) and Claims (i), (ii) we find that, $\max\{B_{11}, B_{12}, \dots, B_{21}\} < B_{11} \leq \frac{B_{19}}{\varepsilon^2}$. So using (9.9) we get each of $B_{10}, B_9, B_8, B_7, B_6$ is $< \frac{B_{19}}{\varepsilon^2}$ and so $B_5 \leq \frac{4}{3}B_6 < \frac{4}{3}\frac{B_{19}}{\varepsilon^2}$ and $B_4 \leq \frac{3}{2}B_6 < \frac{3}{2}\frac{B_{19}}{\varepsilon^2}$. Using (9.9) we have $B_3 \leq \frac{3}{2}\frac{B_{19}}{\varepsilon^2}$ and hence $B_2 \leq \frac{3}{2}\frac{B_{19}}{\varepsilon^2}$.

We have now

$$\begin{aligned}\frac{1}{2} &= \lambda_3^{(21)} < B_1 B_2 B_3 < \mu_3^{(21)} = (3.6422432) \left(\frac{3}{2}\frac{0.4}{\varepsilon^2}\right)^2 \text{ and} \\ \frac{1}{4} &= \lambda_4^{(21)} < B_1 B_2 B_3 B_4 < \mu_4^{(21)} = (3.6422432) \left(\frac{3}{2}\frac{0.4}{\varepsilon^2}\right)^3.\end{aligned}$$

Now we find that $\max\{\phi_{s,n-s}(\lambda_s^{(21)}), \phi_{s,n-s}(\mu_s^{(21)})\}$, for $s = 3, 4\} = \phi_{3,18}(\mu_3^{(21)})$, which is $< \omega_{21}$. Therefore using Lemma 10 we have $B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{21}\}$, for $i = 4, 5$. Using it together with (9.9) we get each of B_2, B_3, \dots, B_{10} is $< \max\{B_{11}, B_{12}, \dots, B_{21}\} < B_{11} \leq \frac{B_{19}}{\varepsilon^2}$.

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + 2B_7 \cdots + 2B_{19} + 2B_{21} < \{3(\frac{1}{\varepsilon^2}) + 2(4 \times \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)\}B_{19} + 2B_{21} < 28.605$ for $B_{21} < 0.2938$ and $B_{19} < 0.4$, giving thereby a contradiction to the weak inequality $(2, 1, 2, 2, \dots, 2)_w$. \square

9.7 $n = 22$

Here we have $\omega_{22} = 30.62$, $B_1 \leq \gamma_{22} < 3.7640371$. Using (2.5), we have $l_{22} = 0.1655 < B_{22} < 3.4344103 = m_{22}$.

Claim(i) $B_{22} < 0.295$

Suppose $B_{22} \geq 0.295$. The inequality $(22^*, 1)$ gives $28.605(B_{22})^{\frac{-1}{21}} + B_{22} > 30.62$. But this is not true for $0.295 \leq B_{22} \leq 3.4344103$.

Claim(ii) $B_{20} < 0.378$

Suppose $B_{20} \geq 0.378$. Then $B_{21} B_{22} \leq \frac{4}{3} B_{22}^2 < \frac{4}{3} (0.295)^2 < B_{20}^2$. Therefore the inequality $(19^*, 3)$ holds, i.e. $24.691(B_{20} B_{21} B_{22})^{\frac{-1}{19}} + 4B_{20} - \frac{B_{20}^3}{B_{21} B_{22}} > 30.62$. But this is not true for $0.378 \leq B_{20} \leq \frac{3}{2} B_{22} < 0.4425$ and $\frac{1}{2} B_{20}^2 \leq B_{21} B_{22} < \frac{4}{3} (0.295)^2$. So we must have $B_{20} < 0.378$.

Claim(iii) B_2, B_3, \dots, B_{11} is $< \frac{B_{20}}{\varepsilon^2}$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(22)} < B_1 < \mu_1^{(22)} = 3.7640371, \\
\frac{3}{4} &= \lambda_2^{(22)} < B_1 B_2 < \mu_2^{(22)} = (1.1362692)^{20}, \\
\frac{1}{2} &= \lambda_3^{(22)} < B_1 B_2 B_3 < \mu_3^{(22)} = (1.21407992)^{19}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(22)} < B_1 \cdots B_5 < \mu_5^{(22)} = (1.39334)^{17}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(22)} < B_1 \cdots B_6 < \mu_6^{(22)} = (1.496951)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{22}^{15}} &= \lambda_7^{(22)} < B_1 \cdots B_7 < \mu_7^{(22)} = (1.611638)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{22}^{14}} &= \lambda_8^{(22)} < B_1 \cdots B_8 < \mu_8^{(22)} = (1.739055)^{14}, \\
\frac{\varepsilon^{15}}{4^3 B_{22}^{13}} &= \lambda_9^{(22)} < B_1 \cdots B_9 < \mu_9^{(22)} = (1.8811357)^{13}, \\
\frac{\varepsilon^{12}}{4^3 B_{22}^{12}} &= \lambda_{10}^{(22)} < B_1 \cdots B_{10} < \mu_{10}^{(22)} = (2.0402387)^{12}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(22)}), \phi_{s,n-s}(\mu_s^{(22)})\}$, for $s = 1, 2, 3, 5, 6, 7, 8, 9, 10\} = \phi_{1,21}(\mu_1^{(22)})$, which is $< \omega_{22}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{22}\}, \text{ for } i = 2, 3, 4, 6, 7, 8, 9, 10, 11. \quad (9.10)$$

From (2.1), (2.2) and Claims(i), (ii) we find that, $\max\{B_{12}, B_{13}, \dots, B_{22}\} < B_{12} \leq \frac{B_{20}}{\varepsilon^2}$. So using (9.10) we get each of $B_{11}, B_{10}, B_9, B_8, B_7, B_6$ is $< \frac{B_{20}}{\varepsilon^2}$ and so $B_5 \leq \frac{4}{3} B_6 < \frac{4}{3} \frac{B_{20}}{\varepsilon^2}$. Again using (9.10) we get each of B_4, B_3 and B_2 is $< \frac{4}{3} \frac{B_{20}}{\varepsilon^2}$.

We have now

$$\frac{1}{4} = \lambda_4^{(22)} < B_1 B_2 B_3 B_4 < \mu_4^{(22)} = (3.7640371) \left(\frac{4}{3} \frac{0.378}{\varepsilon^2}\right)^3.$$

We find that $\max\{\phi_{4,18}(\lambda_4^{(22)}), \phi_{4,18}(\mu_4^{(22)})\} = \phi_{4,18}(\mu_4^{(22)})$, which is $< \omega_{22}$. Therefore using Lemma 10 we have $B_5 < \max\{B_6, B_7, \dots, B_{22}\}$. Using it together with (9.10) we get each of B_2, B_3, \dots, B_{11} is $< \frac{B_{20}}{\varepsilon^2}$.

Final Contradiction

Now $2B_2 + 2B_4 + \dots + 2B_{20} + 2B_{22} < 2\{6 \times \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\} B_{20} + 2B_{22} < 30.62$ for $B_{22} < 0.295$ and $B_{20} < 0.378$, giving thereby a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.8 $n = 23$

Here we have $\omega_{23} = 32.68$, $B_1 \leq \gamma_{23} < 3.8854763$. Using (2.5), we have $l_{23} = 0.1556 < B_{23} < 3.5532476 = m_{22}$.

Claim(i) $B_{23} < 0.293$

Suppose $B_{23} \geq 0.293$. The inequality $(22^*, 1)$ gives $30.62(B_{23})^{\frac{-1}{22}} + B_{23} > 32.68$. But this is not true for $0.293 \leq B_{23} < 3.5532476$. So we must have $B_{23} < 0.293$.

Claim(ii) $B_{21} < 0.376$

Suppose $B_{21} \geq 0.376$. The inequality $(20^*, 3)$ gives $26.629(B_{21} B_{22} B_{23})^{\frac{-1}{20}} + 4B_{21} - \frac{B_{21}^3}{B_{22} B_{23}} > 32.68$. But this is not true for $0.376 \leq B_{21} \leq \frac{3}{2} B_{23} < 0.4395$

and $\frac{1}{2}B_{21}^2 \leq B_{22}B_{23} \leq \frac{4}{3}B_{22}^2 < \frac{4}{3}(0.293)^2$. So we must have $B_{21} < 0.376$.

Claim(iii) $B_2, \dots, B_{11} < \max\{B_{12}, \frac{B_{21}}{\varepsilon^2}\}$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(23)} < B_1 < \mu_1^{(23)} = 3.8854763, \\ \frac{3}{4} &= \lambda_2^{(23)} < B_1B_2 < \mu_2^{(23)} = (1.1329363)^{21}, \\ \frac{1}{2} &= \lambda_3^{(23)} < B_1B_2B_3 < \mu_3^{(23)} = (1.2085769)^{20}, \\ \frac{1}{4} &= \lambda_4^{(23)} < B_1B_2B_3B_4 < \mu_4^{(23)} = (1.2913392)^{19}, \\ \frac{\varepsilon}{4} &= \lambda_5^{(23)} < B_1 \cdots B_5 < \mu_5^{(23)} = (1.3821298)^{18}, \\ \frac{3\varepsilon^2}{16} &= \lambda_6^{(23)} < B_1 \cdots B_6 < \mu_6^{(23)} = (1.4820067)^{17}, \\ \frac{\varepsilon^3}{8} &= \lambda_7^{(23)} < B_1 \cdots B_7 < \mu_7^{(23)} = (1.5922103)^{16}, \\ \frac{\varepsilon^4}{16} &= \lambda_8^{(23)} < B_1 \cdots B_8 < \mu_8^{(23)} = (1.7142025)^{15}, \\ \frac{3\varepsilon^{18}}{4^4B_{23}^{14}} &= \lambda_9^{(23)} < B_1 \cdots B_9 < \mu_9^{(23)} = (1.8497215)^{14}, \\ \frac{\varepsilon^{15}}{4^3B_{23}^{13}} &= \lambda_{10}^{(23)} < B_1 \cdots B_{10} < \mu_{10}^{(23)} = (2.000844)^{13}, \\ \frac{\varepsilon^{12}}{4^3B_{23}^{12}} &= \lambda_{11}^{(23)} < B_1 \cdots B_{11} < \mu_{11}^{(23)} = (2.1700714)^{12}, \\ \frac{\varepsilon^{10}}{2 \times 4^2B_{23}^{11}} &= \lambda_{12}^{(23)} < B_1 \cdots B_{12} < \mu_{12}^{(23)} = (2.3604401)^{11}. \end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(23)}), \phi_{s,n-s}(\mu_s^{(23)})$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \phi_{1,22}(\mu_1^{(23)})$, which is $< \omega_{23}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{23}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. \quad (9.11)$$

From (2.1),(2.2) and Claims (i),(ii) we find that, $\max\{B_{13}, B_{14}, \dots, B_{23}\} < B_{13} \leq \frac{B_{21}}{\varepsilon^2} < 1.7114$. So using (9.11) we get each of B_2, \dots, B_{11} is $< \max\{B_{12}, \frac{B_{21}}{\varepsilon^2}\}$.

Claim(iv) $B_{12} < \frac{B_{21}}{\varepsilon^2}$

We find that $\phi_{11,12}(\lambda_{11}^{(23)}) < \omega_{23}$, but $\phi_{11,12}(\mu_{11}^{(23)}) > \omega_{23}$, so we apply Lemma 11 with $\sigma_{11}^{(23)} = (2.135)^{12}$. Here $\phi_{11,12}(\sigma_{11}^{(23)}) < \omega_{23}$.

In Case(i), when $B_1B_2 \cdots B_{11} < (2.135)^{12}$, then we have $B_{12} < \max\{B_{13}, B_{14}, \dots, B_{23}\}$, which is $\leq \frac{B_{21}}{\varepsilon^2} < 1.7114$.

In Case(ii), when $B_1B_2 \cdots B_{11} \geq (2.135)^{12}$, then we have $B_{12} < \frac{\mu_{12}^{(23)}}{\sigma_{11}^{(23)}} <$

$$\frac{(2.3604)^{11}}{(2.135)^{12}} < 1.43.$$

So we have $B_{12} < \frac{B_{21}}{\varepsilon^2}$.

Using Claims(iii), (iv) we get, each of B_2, B_3, \dots, B_{13} is $< \frac{B_{21}}{\varepsilon^2}$.

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + 2B_7 + \cdots + 2B_{23} < 3(\frac{B_{21}}{\varepsilon^2}) + 2\{5(\frac{1}{\varepsilon^2}) + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\}B_{21} + 2B_{23} < 32.68$ for $B_{21} < 0.376$ and $B_{23} < 0.293$, giving thereby a contradiction to the weak inequality $(2, 1, 2, 2, \dots, 2, 2)_w$. \square

9.9 $n = 24$

Here we have $\omega_{24} = 34.78$, $B_1 \leq \gamma_{24} < 4.0065998$. Using (2.5), we have $l_{24} = 0.1464 < B_{24} < 3.6718429 = m_{24}$.

Claim(i) $B_{24} < 0.29$

Suppose $B_{24} \geq 0.29$. The inequality $(23^*, 1)$ gives $32.68(B_{24})^{\frac{-1}{23}} + B_{24} > 34.78$. But this is not true for $0.29 \leq B_{24} < 3.6718429$. So we must have $B_{24} < 0.29$.

Claim(ii) $B_{22} < 0.374$

Suppose $B_{22} \geq 0.374$. The inequality $(21^*, 3)$ gives $28.605(B_{22}B_{23}B_{24})^{\frac{-1}{21}} + 4B_{22} - \frac{B_{22}^3}{B_{23}B_{24}} > 34.78$. But this is not true for $0.374 \leq B_{22} \leq \frac{3}{2}B_{24} < 0.435$ and $\frac{1}{2}B_{22}^2 \leq B_{23}B_{24} \leq \frac{4}{3}B_{24}^2 < \frac{4}{3}(0.29)^2$. So we must have $B_{22} < 0.374$.

Claim(iii) $B_i < \max\{B_{12}, B_{13}, 1.703\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(24)} < B_1 < \mu_1^{(24)} = 4.0065998, \\ \frac{3}{4} &= \lambda_2^{(24)} < B_1B_2 < \mu_2^{(24)} = (1.1298027)^{22}, \\ \frac{1}{2} &= \lambda_3^{(24)} < B_1B_2B_3 < \mu_3^{(24)} = (1.2034139)^{21}, \\ \frac{1}{4} &= \lambda_4^{(24)} < B_1B_2B_3B_4 < \mu_4^{(24)} = (1.2837599)^{20}, \\ \frac{\varepsilon}{4} &= \lambda_5^{(24)} < B_1 \cdots B_5 < \mu_5^{(24)} = (1.3716707)^{19}, \\ \frac{3\varepsilon^2}{16} &= \lambda_6^{(24)} < B_1 \cdots B_6 < \mu_6^{(24)} = (1.4681092)^{18}, \\ \frac{\varepsilon^3}{8} &= \lambda_7^{(24)} < B_1 \cdots B_7 < \mu_7^{(24)} = (1.5741992)^{17}, \\ \frac{\varepsilon^4}{16} &= \lambda_8^{(24)} < B_1 \cdots B_8 < \mu_8^{(24)} = (1.6912584)^{16}, \\ \frac{\varepsilon^{21}}{2 \times 4^3 B_{24}^{15}} &= \lambda_9^{(24)} < B_1 \cdots B_9 < \mu_9^{(24)} = (1.8208394)^{15}, \\ \frac{3\varepsilon^{18}}{4^4 B_{24}^{14}} &= \lambda_{10}^{(24)} < B_1 \cdots B_{10} < \mu_{10}^{(24)} = (1.9647888)^{14}, \\ \frac{\varepsilon^{15}}{4^3 B_{24}^{13}} &= \lambda_{11}^{(24)} < B_1 \cdots B_{11} < \mu_{11}^{(24)} = (2.1253121)^{13}, \\ \frac{\varepsilon^{12}}{4^3 B_{24}^{12}} &= \lambda_{12}^{(24)} < B_1 \cdots B_{12} < \mu_{12}^{(24)} = (2.3050671)^{12}, \\ \frac{\varepsilon^{10}}{2 \times 4^2 B_{24}^{11}} &= \lambda_{13}^{(24)} < B_1 \cdots B_{13} < \mu_{13}^{(24)} = (2.5072781)^{11}. \end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(24)}), \phi_{s,n-s}(\mu_s^{(24)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \phi_{1,23}(\mu_1^{(24)})$, which is $< \omega_{24}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{24}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. \quad (9.12)$$

From (2.1), (2.2) and Claims (i), (ii), we find that $\max\{B_{14}, B_{15}, \dots, B_{24}\} < B_{14} \leq \frac{B_{22}}{\varepsilon^2} < 1.703$. So using (9.12) we get each of B_2, \dots, B_{11} is $< \max\{B_{12}, B_{13}, \frac{B_{22}}{\varepsilon^2}\} < \max\{B_{12}, B_{13}, 1.703\}$.

Claim(iv) $B_{12}, B_{13} < 1.72$

We find that for $s = 12, 11$, $\phi_{s,n-s}(\lambda_s^{(24)}) < \omega_{24}$, but $\phi_{s,n-s}(\mu_s^{(24)}) > \omega_{24}$, so we apply Lemma 11 respectively with $\sigma_{12}^{(24)} = (2.22)^{12}$ and $\sigma_{11}^{(24)} = (2.09)^{13}$.

Here $\phi_{12,12}(\sigma_{12}^{(24)}) < \omega_{24}$ and $\phi_{11,13}(\sigma_{11}^{(24)}) < \omega_{24}$.

First consider Lemma 11 for $s = 12$ and with $\sigma_{12}^{(24)} = (2.22)^{12}$.

In Case(i), when $B_1 B_2 \cdots B_{12} < (2.22)^{12}$, then we have $B_{13} < \max\{B_{14}, B_{15}, \dots, B_{24}\}$, which is $\leq \frac{B_{22}}{\varepsilon^2} < 1.703$.

In Case(ii), when $B_1 B_2 \cdots B_{12} \geq (2.22)^{12}$, then we have $B_{13} < \frac{\mu_{13}^{(24)}}{\sigma_{12}^{(24)}} < \frac{(2.50718)^{11}}{(2.22)^{12}} < 1.72$.

So we have $B_{13} < 1.72$.

Now consider Lemma 11 for $s = 11$ and with $\sigma_{11}^{(24)} = (2.09)^{13}$.

In Case(i), when $B_1 B_2 \cdots B_{11} < (2.09)^{13}$, then we have $B_{12} < \max\{B_{13}, B_{14}, \dots, B_{24}\}$, which is $< B_{13} < 1.72$.

In Case(ii), when $B_1 B_2 \cdots B_{11} \geq (2.09)^{13}$, then we have $B_{12} < \frac{\mu_{12}^{(24)}}{\sigma_{11}^{(24)}} < \frac{(2.305)^{12}}{(2.09)^{13}} < 1.56$.

Using Claim(iii) and (iv) we get B_2, B_3, \dots, B_{13} is < 1.72 .

Final Contradiction

Now $2B_2 + 2B_4 + \dots + 2B_{24} < 2(6 \times 1.72) + 2\{\frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\}B_{22} + 2B_{24} < 34.78$ for $B_{22} < 0.374$ and $B_{24} < 0.29$, giving thereby a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.10 $n = 25$

Here we have $\omega_{25} = 37.05$, $B_1 \leq \gamma_{25} < 4.1274438$. Using (2.5), we have $l_{25} = 0.1380 < B_{25} < 3.7902246 = m_{25}$.

Claim(i) $B_{25} < 0.26$

Suppose $B_{25} \geq 0.26$. The inequality $(24^*, 1)$ gives $34.78(B_{25})^{\frac{-1}{24}} + B_{25} > 37.05$. But this is not true for $0.26 \leq B_{25} \leq 3.7902246$. So we must have $B_{25} < 0.26$.

Claim(ii) $B_{24} < 0.311$

Suppose $B_{24} \geq 0.311$. The inequality $(23^*, 2)$ gives $32.68(B_{24}B_{25})^{\frac{-1}{23}} + 4B_{24} - \frac{2B_{24}^2}{B_{25}} > 37.05$. But this is not true for $0.311 \leq B_{24} \leq \frac{4}{3}B_{25} < 0.347$ and $\frac{3}{4}(B_{24}) \leq B_{25} < 0.26$. So we must have $B_{24} < 0.311$.

Claim(iii) $B_{23} < 0.3595$

Suppose $B_{23} \geq 0.3595$. The inequality $(22^*, 3)$ gives $30.62(B_{23}B_{24}B_{25})^{\frac{-1}{22}} + 4B_{23} - \frac{B_{23}^3}{B_{24}B_{25}} > 37.05$. But this is not true for $0.3595 \leq B_{23} \leq \frac{3}{2}B_{25} < 0.39$ and $\frac{1}{2}B_{23}^2 \leq B_{24}B_{25} < (0.311)(0.26)$. So we must have $B_{23} < 0.3595$.

Claim(iv) $B_i < \max\{B_{12}, B_{13}, 2.124\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

Using (2.4) and Lemmas 8, 9 we have

$$1 = \lambda_1^{(25)} < B_1 < \mu_1^{(25)} = 4.1274438,$$

$$\begin{aligned}
\frac{3}{4} &= \lambda_2^{(25)} < B_1 B_2 < \mu_2^{(25)} = (1.1268424)^{23}, \\
\frac{1}{2} &= \lambda_3^{(25)} < B_1 B_2 B_3 < \mu_3^{(25)} = (1.1985502)^{22}, \\
\frac{1}{4} &= \lambda_4^{(25)} < B_1 B_2 B_3 B_4 < \mu_4^{(25)} = (1.2766406)^{21}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(25)} < B_1 \cdots B_5 < \mu_5^{(25)} = (1.3618756)^{20}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(25)} < B_1 \cdots B_6 < \mu_6^{(25)} = (1.4551357)^{19}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(25)} < B_1 \cdots B_7 < \mu_7^{(25)} = (1.5574424)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{25}^{17}} &= \lambda_8^{(25)} < B_1 \cdots B_8 < \mu_8^{(25)} = (1.669988)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{25}^{16}} &= \lambda_9^{(25)} < B_1 \cdots B_9 < \mu_9^{(25)} = (1.794170)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{25}^{15}} &= \lambda_{10}^{(25)} < B_1 \cdots B_{10} < \mu_{10}^{(25)} = (1.9316359)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{25}^{14}} &= \lambda_{11}^{(25)} < B_1 \cdots B_{11} < \mu_{11}^{(25)} = (2.0843445)^{14}, \\
\frac{\varepsilon^{15}}{4^3 B_{25}^{13}} &= \lambda_{12}^{(25)} < B_1 \cdots B_{12} < \mu_{12}^{(25)} = (2.2546355)^{13}, \\
\frac{\varepsilon^{12}}{4^3 B_{25}^{12}} &= \lambda_{13}^{(25)} < B_1 \cdots B_{13} < \mu_{13}^{(25)} = (2.4453284)^{12}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(25)}), \phi_{s,n-s}(\mu_s^{(25)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \phi_{1,24}(\mu_1^{(25)})$, which is $< \omega_{25}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{25}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. \quad (9.13)$$

From (2.1), (2.2) and Claims(i), (ii) and (iii) we find that, $\max\{B_{14}, B_{15}, \dots, B_{25}\} < B_{14} \leq \frac{3}{2} \frac{B_{24}}{\varepsilon^2} < 2.124$. So using (9.13) we get each of B_2, \dots, B_{11} is $< \max\{B_{12}, B_{13}, \frac{3}{2} \frac{B_{24}}{\varepsilon^2}\} < \max\{B_{12}, B_{13}, 2.124\}$.

Claim(v) $B_{12}, B_{13} < 2.124$

Now we find that for $s = 12, 11$, $\phi_{s,n-s}(\lambda_s^{(25)}) < \omega_{25}$, but $\phi_{s,n-s}(\mu_s^{(25)}) > \omega_{25}$, so we apply Lemma 11 respectively with $\sigma_{12}^{(25)} = (2.193)^{13}$ and $\sigma_{11}^{(25)} = (2.07)^{14}$. Here $\phi_{12,13}(\sigma_{12}^{(25)}) < \omega_{25}$ and $\phi_{11,14}(\sigma_{11}^{(25)}) < \omega_{25}$.

First consider Lemma 11 for $s = 12$ and with $\sigma_{12}^{(25)} = (2.193)^{13}$.

In Case(i), when $B_1 B_2 \cdots B_{12} < (2.193)^{13}$, then we have $B_{13} < \max\{B_{14}, B_{15}, \dots, B_{25}\}$, which is $\leq \frac{3}{2} \frac{B_{24}}{\varepsilon^2} < 2.124$.

In Case(ii), when $B_1 B_2 \cdots B_{12} \geq (2.193)^{13}$, then we have $B_{13} < \frac{\mu_{13}^{(25)}}{\sigma_{12}^{(25)}} < \frac{(2.4453)^{12}}{(2.193)^{13}} < 1.686$.

So we have $B_{13} \leq \frac{3}{2} \frac{B_{24}}{\varepsilon^2} < 2.124$.

Now consider Lemma 11 for $s = 11$ and with $\sigma_{11}^{(25)} = (2.07)^{14}$.

In Case(i), when $B_1 B_2 \cdots B_{11} < (2.07)^{14}$, then we have $B_{12} < \max\{B_{13}, B_{14}, \dots, B_{25}\}$, which is $< B_{13} \leq \frac{3}{2} \frac{B_{24}}{\varepsilon^2} < 2.124$.

In Case(ii), when $B_1 B_2 \cdots B_{11} \geq (2.07)^{14}$, then we have $B_{12} < \frac{\mu_{12}^{(25)}}{\sigma_{11}^{(25)}} < \frac{(2.255)^{13}}{(2.07)^{14}} < 1.471$.

So we have $B_{12}, B_{13} < 2.124$.

Using Claim(iii) and (iv) we get each of B_2, B_3, \dots, B_{14} is $< B_{13} < 2.124$.

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + \cdots + 2B_{25} < 3(2.124) + 2(5 \times 2.124) + 2\{(\frac{1}{\varepsilon^2}) + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\}B_{23} + 2B_{25} < 37.05$ for $B_{25} < 0.26$ and $B_{23} < 0.3595$, giving thereby a contradiction to the weak inequality $(2, 1, 2, \dots, 2, 2)_w$. \square

9.11 $n = 26$

Here we have $\omega_{26} = 39.24$, $B_1 \leq \gamma_{26} < 4.2480446$. Using (2.5), we have $l_{26} = 0.1303 < B_{26} < 3.9084192 = m_{26}$.

Claim(i) $B_{26} < 0.29$

Suppose $B_{26} \geq 0.29$. The inequality $(25^*, 1)$ gives $37.05(B_{26})^{\frac{-1}{25}} + B_{26} > 39.24$. But this is not true for $0.29 \leq B_{26} < 3.9084192$.

Claim(ii) $B_{25} < 0.31$

Suppose $B_{25} \geq 0.31$. Then $2B_{25} > B_{26}$. Therefore the inequality $(24^*, 2)$ holds, i.e. $34.78(B_{25}B_{26})^{\frac{-1}{24}} + 4B_{25} - \frac{2B_{25}^2}{B_{26}} > 39.24$. But this is not true for $0.31 \leq B_{25} \leq \frac{4}{3}B_{26} < 0.387$ and $\frac{3}{4}(B_{25}) \leq B_{26} < 0.29$. So we must have $B_{25} < 0.31$.

Claim(iii) $B_{24} < 0.358$

Suppose $B_{24} \geq 0.358$. Then $B_{24}^2 > B_{25}B_{26}$. Therefore the inequality $(23^*, 3)$ holds, i.e. $32.68(B_{24}B_{25}B_{26})^{\frac{-1}{23}} + 4B_{24} - \frac{B_{24}^3}{B_{25}B_{26}} > 39.24$. But this is not true for $0.358 \leq B_{24} \leq \frac{3}{2}B_{26} < 0.436$ and $\frac{1}{2}B_{24}^2 \leq B_{25}B_{26} < (0.31)(0.29)$. So we must have $B_{24} < 0.358$.

Claim(iv) $B_i < \max\{B_{12}, B_{13}, B_{14}, 2.1165\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(26)} < B_1 < \mu_1^{(26)} = 4.2480446, \\
\frac{3}{4} &= \lambda_2^{(26)} < B_1B_2 < \mu_2^{(26)} = (1.1240391)^{24}, \\
\frac{1}{2} &= \lambda_3^{(26)} < B_1B_2B_3 < \mu_3^{(26)} = (1.1939633)^{23}, \\
\frac{1}{4} &= \lambda_4^{(26)} < B_1B_2B_3B_4 < \mu_4^{(26)} = (1.2699424)^{22}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(26)} < B_1 \cdots B_5 < \mu_5^{(26)} = (1.3526843)^{21}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(26)} < B_1 \cdots B_6 < \mu_6^{(26)} = (1.4429963)^{20}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(26)} < B_1 \cdots B_7 < \mu_7^{(26)} = (1.5418115)^{19}, \\
\frac{\varepsilon^4}{16} &= \lambda_8^{(26)} < B_1 \cdots B_8 < \mu_8^{(26)} = (1.650213)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{26}^{17}} &= \lambda_9^{(26)} < B_1 \cdots B_9 < \mu_9^{(24)} = (1.7694615)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{26}^{16}} &= \lambda_{10}^{(26)} < B_1 \cdots B_{10} < \mu_{10}^{(26)} = (1.9010405)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{26}^{15}} &= \lambda_{11}^{(26)} < B_1 \cdots B_{11} < \mu_{11}^{(26)} = (2.0466947)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{26}^{14}} &= \lambda_{12}^{(26)} < B_1 \cdots B_{12} < \mu_{12}^{(26)} = (2.2084995)^{14}, \\
\frac{\varepsilon^{15}}{4^3 B_{26}^{13}} &= \lambda_{13}^{(26)} < B_1 \cdots B_{13} < \mu_{13}^{(26)} = (2.3889339)^{13},
\end{aligned}$$

$$\frac{\varepsilon^{12}}{4^3 B_{26}^{12}} = \lambda_{14}^{(26)} < B_1 \cdots B_{14} < \mu_{14}^{(26)} = (2.5909855)^{12}.$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(26)}), \phi_{s,n-s}(\mu_s^{(26)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \phi_{1,25}(\mu_1^{(26)})$, which is $< \omega_{26}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{26}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. \quad (9.14)$$

From (2.1), (2.2) and Claims(i), (ii), (iii), we find $\max\{B_{15}, B_{16}, \dots, B_{26}\} < B_{15} \leq \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$. So using (9.14) we get that each of B_2, \dots, B_{11} is $< \max\{B_{12}, B_{13}, B_{14}, \frac{3}{2} \frac{B_{25}}{\varepsilon^2}\} < \max\{B_{12}, B_{13}, B_{14}, 2.1165\}$.

Claim(v) $B_{12}, B_{13}, B_{14} < 2.1165$

We find that for $s = 13, 12, 11$, $\phi_{s,n-s}(\lambda_s^{(26)}) < \omega_{26}$, but $\phi_{s,n-s}(\mu_s^{(26)}) > \omega_{26}$, so we apply Lemma 11 respectively with $\sigma_{13}^{(26)} = (2.274)^{13}$, $\sigma_{12}^{(26)} = (2.15)^{14}$ and $\sigma_{11}^{(26)} = (2.036)^{15}$. Here $\phi_{13,13}(\sigma_{13}^{(26)}) < \omega_{26}$, $\phi_{12,14}(\sigma_{12}^{(26)}) < \omega_{26}$ and $\phi_{11,15}(\sigma_{11}^{(26)}) < \omega_{26}$.

First consider Lemma 11 for $s = 13$ and with $\sigma_{13}^{(26)} = (2.274)^{13}$.

In Case(i), when $B_1 B_2 \cdots B_{13} < (2.274)^{13}$, then we have $B_{14} < \max\{B_{15}, B_{16}, \dots, B_{26}\}$, which is $< \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$.

In Case(ii), when $B_1 B_2 \cdots B_{13} \geq (2.274)^{13}$, then we have $B_{14} < \frac{\mu_{14}^{(26)}}{\sigma_{13}^{(26)}} < \frac{(2.591)^{12}}{(2.274)^{13}} < 2.1055$.

So we have $B_{14} < \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$.

Now consider Lemma 11 for $s = 12$ and with $\sigma_{12}^{(26)} = (2.15)^{14}$.

In Case(i), when $B_1 B_2 \cdots B_{12} < (2.15)^{14}$, then we have $B_{13} < \max\{B_{14}, B_{15}, \dots, B_{26}\}$, which is $< \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$.

In Case(ii), when $B_1 B_2 \cdots B_{12} \geq (2.15)^{14}$, then we have $B_{13} < \frac{\mu_{13}^{(26)}}{\sigma_{12}^{(26)}} < \frac{(2.389)^{13}}{(2.15)^{14}} < 1.831$.

So we have $B_{13} < \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$.

Now consider Lemma 11 for $s = 11$ and with $\sigma_{11}^{(26)} = (2.036)^{15}$.

In Case(i), when $B_1 B_2 \cdots B_{11} < (2.036)^{15}$, then we have $B_{12} < \max\{B_{13}, B_{14}, \dots, B_{26}\}$, which is $< \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$.

In Case(ii), when $B_1 B_2 \cdots B_{11} \geq (2.036)^{15}$, then we have $B_{12} < \frac{\mu_{12}^{(26)}}{\sigma_{11}^{(26)}} < \frac{(2.2085)^{14}}{(2.036)^{15}} < 1.54$.

So we have $B_{12} < B_{13} < \frac{3}{2} \frac{B_{25}}{\varepsilon^2} < 2.1165$.

Using Claims(iv) and (v) we get each of B_2, B_3, \dots, B_{15} is < 2.1165 .

Final Contradiction

Now $2B_2 + 2B_4 + \dots + 2B_{26} < 2(7 \times 2.1165) + 2\{(\frac{1}{\varepsilon^2}) + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\}B_{24} + 2B_{26} < 39.24$ for $B_{26} < 0.29$ and $B_{24} < 0.358$, giving thereby a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.12 $n = 27$

Here we have $\omega_{27} = 41.78$, $B_1 \leq \gamma_{27} < 4.3684312$. Using (2.5), we have $l_{27} = 0.1231 < B_{27} < 4.0264547 = m_{27}$.

Claim(i) $B_{27} < 0.226$

Suppose $B_{27} \geq 0.226$. The inequality $(26^*, 1)$ gives $39.24(B_{27})^{\frac{-1}{26}} + B_{27} > 41.78$. But this is not true for $0.226 \leq B_{27} \leq 4.0264547$. So we must have $B_{27} < 0.226$.

Claim(ii) $B_i < \max\{B_{13}, B_{14}, 2.195\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(27)} < B_1 < \mu_1^{(27)} = 4.3684312, \\
\frac{3}{4} &= \lambda_2^{(27)} < B_1 B_2 < \mu_2^{(27)} = (1.1213882)^{25}, \\
\frac{1}{2} &= \lambda_3^{(27)} < B_1 B_2 B_3 < \mu_3^{(27)} = (1.1896237)^{24}, \\
\frac{1}{4} &= \lambda_4^{(27)} < B_1 B_2 B_3 B_4 < \mu_4^{(27)} = (1.2636277)^{23}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(27)} < B_1 \cdots B_5 < \mu_5^{(27)} = (1.34403991)^{22}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(27)} < B_1 \cdots B_6 < \mu_6^{(27)} = (1.4316096)^{21}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(27)} < B_1 \cdots B_7 < \mu_7^{(27)} = (1.5271911)^{20}, \\
\frac{\varepsilon^{36}}{2 \times 4^4 B_{27}^{19}} &= \lambda_8^{(27)} < B_1 \cdots B_8 < \mu_8^{(27)} = (1.6317719)^{19}, \\
\frac{3\varepsilon^{32}}{4^5 B_{27}^{18}} &= \lambda_9^{(27)} < B_1 \cdots B_9 < \mu_9^{(27)} = (1.7464974)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{27}^{17}} &= \lambda_{10}^{(27)} < B_1 \cdots B_{10} < \mu_{10}^{(27)} = (1.8727046)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{27}^{16}} &= \lambda_{11}^{(27)} < B_1 \cdots B_{11} < \mu_{11}^{(27)} = (2.0119609)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{27}^{15}} &= \lambda_{12}^{(27)} < B_1 \cdots B_{12} < \mu_{12}^{(27)} = (2.1661136)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{27}^{14}} &= \lambda_{13}^{(27)} < B_1 \cdots B_{13} < \mu_{13}^{(27)} = (2.3373592)^{14}, \\
\frac{\varepsilon^{15}}{4^3 B_{27}^{13}} &= \lambda_{14}^{(27)} < B_1 \cdots B_{14} < \mu_{14}^{(27)} = (2.5283215)^{13}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(27)}), \phi_{s,n-s}(\mu_s^{(27)})$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = \phi_{11,16}(\mu_{11}^{(27)})$, which is $< \omega_{27}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{27}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. \quad (9.15)$$

From (2.1), (2.2) and Claim(i) we find that, $\max\{B_{15}, B_{16}, \dots, B_{27}\} < B_{15} \leq \frac{B_{27}}{\varepsilon^3} < 2.195$. So using (9.15) we get that each of B_2, \dots, B_{12} is $< \max\{B_{13}, B_{14}, \frac{B_{27}}{\varepsilon^3}\} < \max\{B_{13}, B_{14}, 2.195\}$.

Claim(iii) $B_{13}, B_{14} < 2.195$

We find that for $s = 13, 12$, $\phi_{s,n-s}(\lambda_s^{(27)}) < \omega_{27}$, but $\phi_{s,n-s}(\mu_s^{(27)}) > \omega_{27}$, so we apply Lemma 11 respectively with $\sigma_{13}^{(27)} = (2.25)^{14}$ and $\sigma_{12}^{(27)} = (2.13)^{15}$.

Here $\phi_{13,14}(\sigma_{13}^{(27)}) < \omega_{27}$ and $\phi_{12,15}(\sigma_{12}^{(27)}) < \omega_{27}$.

First consider Lemma 11 for $s = 13$ and with $\sigma_{13}^{(27)} = (2.25)^{14}$.

In Case(i), when $B_1 B_2 \cdots B_{13} < (2.25)^{14}$, then we have $B_{14} < \max\{B_{15}, B_{16}, \dots, B_{27}\}$,

which is $< \frac{B_{27}}{\varepsilon^3} < 2.195$.

In Case(ii), when $B_1 B_2 \cdots B_{13} \geq (2.25)^{14}$, then we have $B_{14} < \frac{\mu_{14}^{(27)}}{\sigma_{13}^{(27)}} < \frac{(2.5284)^{13}}{(2.25)^{14}} < 2.026$.

So we have $B_{14} < \frac{B_{27}}{\varepsilon^3} < 2.195$.

Now consider Lemma 11 for $s = 12$ and with $\sigma_{12}^{(27)} = (2.13)^{15}$.

In Case(i), when $B_1 B_2 \cdots B_{12} < (2.13)^{15}$, then we have $B_{13} < \max\{B_{14}, B_{15}, \dots, B_{27}\}$, which is $< \frac{B_{27}}{\varepsilon^3} < 2.195$.

In Case(ii), when $B_1 B_2 \cdots B_{12} \geq (2.13)^{15}$, then we have $B_{13} < \frac{\mu_{13}^{(27)}}{\sigma_{12}^{(27)}} < \frac{(2.3374)^{14}}{(2.13)^{15}} < 1.725$.

So we have $B_{13} < \frac{B_{27}}{\varepsilon^3} < 2.195$.

Using Claim(ii) and (iii) we get each of B_2, B_3, \dots, B_{15} is < 2.195 .

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + \cdots + 2B_{27} < 3(\frac{B_{27}}{\varepsilon^3}) + 2\{(6 \times \frac{1}{\varepsilon^3} + \frac{3/2}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)\}B_{27} < 41.78$ for $B_{27} < 0.226$, giving thereby a contradiction to the weak inequality $(2, 1, 2, \dots, 2, 2)_w$. \square

9.13 $n = 28$

Here we have $\omega_{28} = 44.36$, $B_1 \leq \gamma_{28} < 4.488631$. Using (2.5), we have $l_{28} = 0.1164 < B_{28} < 4.144353 = m_{28}$.

Claim(i) $B_{28} < 0.228$

Suppose $B_{28} \geq 0.228$. The inequality $(27^*, 1)$ gives $41.78(B_{28})^{\frac{-1}{27}} + B_{28} > 44.36$. But this is not true for $0.228 \leq B_{28} \leq 4.144353$. So we must have $B_{28} < 0.228$.

Claim(ii) $B_i < \max\{B_{13}, B_{14}, B_{15}, 2.215\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(28)} < B_1 < \mu_1^{(28)} = 4.488631, \\ \frac{3}{4} &= \lambda_2^{(28)} < B_1 B_2 < \mu_2^{(28)} = (1.118873)^{26}, \\ \frac{1}{2} &= \lambda_3^{(28)} < B_1 B_2 B_3 < \mu_3^{(28)} = (1.1855192)^{25}, \\ \frac{1}{4} &= \lambda_4^{(28)} < B_1 B_2 B_3 B_4 < \mu_4^{(28)} = (1.257657)^{24}, \\ \frac{\varepsilon}{4} &= \lambda_5^{(28)} < B_1 \cdots B_5 < \mu_5^{(28)} = (1.3358932)^{23}, \\ \frac{3\varepsilon^2}{16} &= \lambda_6^{(28)} < B_1 \cdots B_6 < \mu_6^{(28)} = (1.4209042)^{22}, \\ \frac{\varepsilon^3}{8} &= \lambda_7^{(28)} < B_1 \cdots B_7 < \mu_7^{(28)} = (1.5134819)^{21}, \\ \frac{\varepsilon^{40}}{4^5 B_{28}^{20}} &= \lambda_8^{(28)} < B_1 \cdots B_8 < \mu_8^{(28)} = (1.6145296)^{20}, \\ \frac{\varepsilon^{36}}{2 \times 4^4 B_{28}^{19}} &= \lambda_9^{(28)} < B_1 \cdots B_9 < \mu_9^{(24)} = (1.7250912)^{19}, \\ \frac{3\varepsilon^{32}}{4^5 B_{28}^{18}} &= \lambda_{10}^{(28)} < B_1 \cdots B_{10} < \mu_{10}^{(28)} = (1.8463778)^{18}, \end{aligned}$$

$$\begin{aligned}
\frac{\varepsilon^{28}}{4^4 B_{28}^{17}} &= \lambda_{11}^{(28)} < B_1 \cdots B_{11} < \mu_{11}^{(28)} = (1.9798026)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{28}^{16}} &= \lambda_{12}^{(28)} < B_1 \cdots B_{12} < \mu_{12}^{(28)} = (2.1270228)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{28}^{15}} &= \lambda_{13}^{(28)} < B_1 \cdots B_{13} < \mu_{13}^{(28)} = (2.2899914)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{28}^{14}} &= \lambda_{14}^{(28)} < B_1 \cdots B_{14} < \mu_{14}^{(28)} = (2.4710303)^{14}, \\
\frac{\varepsilon^{15}}{4^3 B_{28}^{13}} &= \lambda_{15}^{(28)} < B_1 \cdots B_{15} < \mu_{15}^{(28)} = (2.6729135)^{13}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(28)}), \phi_{s,n-s}(\mu_s^{(28)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\} = \phi_{11,17}(\mu_{11}^{(28)})$, which is $< \omega_{28}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{28}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. \quad (9.16)$$

From (2.1), (2.2) and Claim(i) we find that, $\max\{B_{16}, B_{17}, \dots, B_{28}\} < B_{16} \leq \frac{B_{28}}{\varepsilon^3} < 2.215$. So using (9.16) we get each of B_2, \dots, B_{12} is $< \max\{B_{13}, B_{14}, B_{15}, \frac{B_{28}}{\varepsilon^3}\} < \max\{B_{13}, B_{14}, B_{15}, 2.215\}$.

Claim(iii) $B_{13}, B_{14}, B_{15} < 2.215$

We find that for $s = 14, 13, 12$, $\phi_{s,n-s}(\lambda_s^{(28)}) < \omega_{28}$, but $\phi_{s,n-s}(\mu_s^{(28)}) > \omega_{28}$, so we apply Lemma 11 respectively with $\sigma_{14}^{(28)} = (2.355)^{14}$, $\sigma_{13}^{(28)} = (2.23)^{15}$ and $\sigma_{12}^{(28)} = (2.11)^{16}$. Here $\phi_{14,14}(\sigma_{14}^{(28)}) < \omega_{28}$, $\phi_{13,15}(\sigma_{13}^{(28)}) < \omega_{28}$ and $\phi_{12,16}(\sigma_{12}^{(28)}) < \omega_{28}$.

First consider Lemma 11 for $s = 14$ and with $\sigma_{14}^{(28)} = (2.355)^{14}$.

In Case(i), when $B_1 B_2 \cdots B_{14} < (2.355)^{14}$, then we have $B_{15} < \max\{B_{16}, B_{17}, \dots, B_{28}\}$, which is $< \frac{B_{28}}{\varepsilon^3} < 2.215$.

In Case(ii), when $B_1 B_2 \cdots B_{14} \geq (2.355)^{14}$, then we have $B_{15} < \frac{\mu_{15}^{(28)}}{\sigma_{14}^{(28)}} < \frac{(2.673)^{13}}{(2.355)^{14}} < 2.204$.

So we have $B_{15} < \frac{B_{28}}{\varepsilon^3} < 2.215$.

Now consider Lemma 11 for $s = 13$ and with $\sigma_{13}^{(28)} = (2.23)^{15}$.

In Case(i), when $B_1 B_2 \cdots B_{13} < (2.23)^{15}$, then we have $B_{14} < \max\{B_{15}, B_{16}, \dots, B_{28}\}$, which is $< \frac{B_{28}}{\varepsilon^3} < 2.215$.

In Case(ii), when $B_1 B_2 \cdots B_{13} \geq (2.23)^{15}$, then we have $B_{14} < \frac{\mu_{14}^{(28)}}{\sigma_{13}^{(28)}} < \frac{(2.472)^{14}}{(2.23)^{15}} < 1.898$.

So we have $B_{14} < \frac{B_{28}}{\varepsilon^3} < 2.215$.

Now consider Lemma 11 for $s = 12$ and with $\sigma_{12}^{(28)} = (2.11)^{16}$.

In Case(i), when $B_1 B_2 \cdots B_{12} < (2.11)^{16}$, then we have $B_{13} < \max\{B_{14}, B_{15}, \dots, B_{28}\}$, which is $< \frac{B_{28}}{\varepsilon^3} < 2.215$.

In Case(ii), when $B_1 B_2 \cdots B_{12} \geq (2.11)^{16}$, then we have $B_{13} < \frac{\mu_{13}^{(28)}}{\sigma_{12}^{(28)}} < \frac{(2.2899)^{15}}{(2.11)^{16}} < 1.62$.

So we have $B_{13} < \frac{B_{28}}{\varepsilon^3} < 2.215$.

Using Claim(ii) and (iii) we get each of B_2, B_3, \dots, B_{16} is < 2.215 .

Final Contradiction

Now $2B_2 + 2B_4 + \dots + 2B_{28} < 2\{8 \times \frac{1}{\varepsilon^3} + \frac{3/2}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1\}B_{28} < 44.36$ for $B_{28} < 0.228$, giving thereby a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.14 $n = 29$

Here $\omega_{29} = 47.18$, $B_1 \leq \gamma_{29} < 4.6086676$. Using (2.5), we have $l_{29} = 0.1102 < B_{29} < 4.2621353 = m_{29}$.

Claim(i) $B_{29} < 0.201$

The inequality $(28^*, 1)$ gives $44.36(B_{29})^{\frac{-1}{28}} + B_{29} > 47.18$. But this is not true for $0.201 \leq B_{29} \leq 4.2621353$. So we must have $B_{29} < 0.201$.

Claim(ii) $B_{28} < 0.246$

The inequality $(27^*, 2)$ gives $41.78(B_{28}B_{29})^{\frac{-1}{27}} + 4B_{28} - \frac{2B_{28}^2}{B_{29}} > 47.18$. But this is not true for $0.246 \leq B_{28} \leq \frac{4}{3}B_{29} < 0.268$ and $\frac{3}{4}(B_{28}) \leq B_{29} < 0.201$. So we must have $B_{28} < 0.246$.

Claim(iii) $B_{27} < 0.2835$

The inequality $(26^*, 3)$ gives $39.24(B_{27}B_{28}B_{29})^{\frac{-1}{26}} + 4B_{27} - \frac{B_{27}^3}{B_{28}B_{29}} > 47.18$. But this is not true for $0.2835 \leq B_{27} \leq \frac{3}{2}B_{29} < 0.302$ and $\frac{1}{2}B_{27}^2 \leq B_{28}B_{29} < (0.246)(0.201)$. So we must have $B_{27} < 0.2835$.

Claim(iv) $B_i < \max\{B_{14}, B_{15}, 2.3888\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(29)} < B_1 < \mu_1^{(29)} = 4.6086676, \\
\frac{3}{4} &= \lambda_2^{(29)} < B_1B_2 < \mu_2^{(29)} = (1.1164824)^{27}, \\
\frac{1}{2} &= \lambda_3^{(29)} < B_1B_2B_3 < \mu_3^{(29)} = (1.18162593)^{26}, \\
\frac{1}{4} &= \lambda_4^{(29)} < B_1B_2B_3B_4 < \mu_4^{(29)} = (1.2520101)^{25}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(29)} < B_1 \cdots B_5 < \mu_5^{(29)} = (1.3281938)^{24}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(29)} < B_1 \cdots B_6 < \mu_6^{(29)} = (1.41081792)^{23}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(29)} < B_1 \cdots B_7 < \mu_7^{(29)} = (1.5005968)^{22}, \\
\frac{\varepsilon^4}{16} &= \lambda_8^{(29)} < B_1 \cdots B_8 < \mu_8^{(29)} = (1.5983668)^{21}, \\
\frac{\varepsilon^{40}}{4^5 B_{29}^{20}} &= \lambda_9^{(29)} < B_1 \cdots B_9 < \mu_9^{(24)} = (1.7050819)^{20}, \\
\frac{\varepsilon^{36}}{2 \times 4^4 B_{29}^{19}} &= \lambda_{10}^{(29)} < B_1 \cdots B_{10} < \mu_{10}^{(29)} = (1.8218444)^{19}, \\
\frac{3\varepsilon^{32}}{4^5 B_{29}^{18}} &= \lambda_{11}^{(29)} < B_1 \cdots B_{11} < \mu_{11}^{(29)} = (1.9499335)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{29}^{17}} &= \lambda_{12}^{(29)} < B_1 \cdots B_{12} < \mu_{12}^{(29)} = (2.0908415)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{29}^{16}} &= \lambda_{13}^{(29)} < B_1 \cdots B_{13} < \mu_{13}^{(29)} = (2.2463188)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{29}^{15}} &= \lambda_{14}^{(29)} < B_1 \cdots B_{14} < \mu_{14}^{(29)} = (2.4184275)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{29}^{14}} &= \lambda_{15}^{(29)} < B_1 \cdots B_{15} < \mu_{15}^{(29)} = (2.6096202)^{14}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(29)}), \phi_{s,n-s}(\mu_s^{(29)})$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} = \phi_{12,17}(\mu_{12}^{(29)})$, which is $< \omega_{29}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{29}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. \quad (9.17)$$

From (2.1), (2.2) and Claims (i), (ii) and (iii) we find that, $\max\{B_{16}, B_{17}, \dots, B_{29}\} < B_{16} \leq \frac{B_{28}}{\varepsilon^3} < 2.3888$. So using (9.17) we get that each of B_2, \dots, B_{13} is $< \max\{B_{14}, B_{15}, \frac{B_{28}}{\varepsilon^3}\} < \max\{B_{14}, B_{15}, 2.3888\}$.

Claim(v) $B_{14}, B_{15} < 2.3888$

We find that for $s = 14, 13$, $\phi_{s,n-s}(\lambda_s^{(29)}) < \omega_{29}$, but $\phi_{s,n-s}(\mu_s^{(29)}) > \omega_{29}$, so we apply Lemma 11 respectively with $\sigma_{14}^{(29)} = (2.34)^{15}$ and $\sigma_{13}^{(29)} = (2.21)^{16}$. Here $\phi_{14,15}(\sigma_{14}^{(29)}) < \omega_{29}$ and $\phi_{13,16}(\sigma_{13}^{(29)}) < \omega_{29}$.

First consider Lemma 11 for $s = 14$ and with $\sigma_{14}^{(29)} = (2.34)^{15}$.

In Case(i), when $B_1 B_2 \dots B_{14} < (2.34)^{15}$, then we have $B_{15} < \max\{B_{16}, B_{17}, \dots, B_{29}\}$, which is $< \frac{B_{28}}{\varepsilon^3} < 2.3888$.

In Case(ii), when $B_1 B_2 \dots B_{14} \geq (2.34)^{15}$, then we have $B_{15} < \frac{\mu_{15}^{(29)}}{\sigma_{14}^{(29)}} < \frac{(2.6097)^{14}}{(2.34)^{15}} < 1.969$.

So we have $B_{15} < \frac{B_{28}}{\varepsilon^3} < 2.3888$.

Now consider Lemma 11 for $s = 13$ and with $\sigma_{13}^{(29)} = (2.21)^{16}$.

In Case(i), when $B_1 B_2 \dots B_{13} < (2.21)^{16}$, then we have $B_{14} < \max\{B_{15}, B_{16}, \dots, B_{29}\}$, which is $< \frac{B_{28}}{\varepsilon^3} < 2.3888$.

In Case(ii), when $B_1 B_2 \dots B_{13} \geq (2.21)^{16}$, then we have $B_{14} < \frac{\mu_{14}^{(29)}}{\sigma_{13}^{(29)}} < \frac{(2.4185)^{15}}{(2.21)^{16}} < 1.75$.

So we have $B_{14} < \frac{B_{28}}{\varepsilon^3} < 2.3888$.

Using Claims(iv) and (v) we get each of B_2, B_3, \dots, B_{16} is < 2.3888 .

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + \dots + 2B_{29} < 3(\frac{B_{28}}{\varepsilon^3}) + 2(6 \times \frac{B_{28}}{\varepsilon^3}) + 2(\frac{3/2}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{27} + 2B_{29} < 47.18$ for $B_{29} < 0.201$, $B_{28} < 0.246$ and $B_{27} < 0.2835$, giving thereby a contradiction to the weak inequality $(2, 1, 2, \dots, 2, 2)_w$. \square

9.15 $n = 30$

Here $\omega_{30} = 49.86$. We have $B_1 \leq \gamma_{30} < 4.7285667$. Using (2.5), we have $l_{30} = 0.1045 < B_{30} < 4.3798196 = m_{30}$.

Claim(i) $B_{30} < 0.231$

Suppose $B_{30} \geq 0.231$. The inequality $(29^*, 1)$ gives $47.18(B_{30})^{\frac{-1}{29}} + B_{30} > 49.86$. But this is not true for $0.231 \leq B_{30} \leq 4.3798196$.

Claim(ii) $B_{29} < 0.247$

Suppose $B_{29} \geq 0.247$, then $2B_{29} > B_{30}$. Therefore the inequality $(28^*, 2)$

holds i.e. $44.36(B_{29}B_{30})^{\frac{-1}{28}} + 4B_{29} - \frac{2B_{29}^2}{B_{30}} > 49.86$. But this is not true for $0.247 \leq B_{29} \leq \frac{4}{3}B_{30} < 0.308$ and $\frac{3}{4}(B_{29}) \leq B_{30} < 0.231$. So we must have $B_{29} < 0.247$.

Claim(iii) $B_{28} < 0.285$

Suppose $B_{28} \geq 0.285$, then $B_{28}^2 > B_{29}B_{30}$. Therefore the inequality $(27^*, 3)$ holds, i.e. $41.78(B_{28}B_{29}B_{30})^{\frac{-1}{27}} + 4B_{28} - \frac{B_{28}^3}{B_{29}B_{30}} > 49.86$. But this is not true for $0.285 \leq B_{28} \leq \frac{3}{2}B_{29} < 0.347$ and $\frac{1}{2}B_{28}^2 < B_{29}B_{30} < (0.247)(0.231)$. So we must have $B_{28} < 0.285$.

Claim(iv) $B_i < \max\{B_{14}, B_{15}, B_{16}, 2.3985\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(30)} < B_1 < \mu_1^{(30)} = 4.7285667, \\
\frac{3}{4} &= \lambda_2^{(30)} < B_1B_2 < \mu_2^{(30)} = (1.114207)^{28}, \\
\frac{1}{2} &= \lambda_3^{(30)} < B_1B_2B_3 < \mu_3^{(30)} = (1.1779273)^{27}, \\
\frac{1}{4} &= \lambda_4^{(30)} < B_1B_2B_3B_4 < \mu_4^{(30)} = (1.2466561)^{26}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(30)} < B_1 \cdots B_5 < \mu_5^{(30)} = (1.3209137)^{25}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(30)} < B_1 \cdots B_6 < \mu_6^{(30)} = (1.4012902)^{24}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(30)} < B_1 \cdots B_7 < \mu_7^{(30)} = (1.4884616)^{23}, \\
\frac{\varepsilon^4}{16} &= \lambda_8^{(30)} < B_1 \cdots B_8 < \mu_8^{(30)} = (1.5831813)^{22}, \\
\frac{\varepsilon^6}{16} &= \lambda_9^{(30)} < B_1 \cdots B_9 < \mu_9^{(24)} = (1.6863321)^{21}, \\
\frac{\varepsilon^{40}}{4^5 B_{30}^{20}} &= \lambda_{10}^{(30)} < B_1 \cdots B_{10} < \mu_{10}^{(30)} = (1.7989201)^{20}, \\
\frac{\varepsilon^{36}}{2 \times 4^4 B_{30}^{19}} &= \lambda_{11}^{(30)} < B_1 \cdots B_{11} < \mu_{11}^{(30)} = (1.9221086)^{19}, \\
\frac{3\varepsilon^{32}}{4^5 B_{30}^{18}} &= \lambda_{12}^{(30)} < B_1 \cdots B_{12} < \mu_{12}^{(30)} = (2.057247)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{30}^{17}} &= \lambda_{13}^{(30)} < B_1 \cdots B_{13} < \mu_{13}^{(30)} = (2.2059099)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{30}^{16}} &= \lambda_{14}^{(30)} < B_1 \cdots B_{14} < \mu_{14}^{(30)} = (2.3699437)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{30}^{15}} &= \lambda_{15}^{(30)} < B_1 \cdots B_{15} < \mu_{15}^{(30)} = (2.551525)^{15}, \\
\frac{3\varepsilon^{18}}{4^4 B_{30}^{14}} &= \lambda_{16}^{(30)} < B_1 \cdots B_{16} < \mu_{16}^{(30)} = (2.7532392)^{14}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(30)}), \phi_{s,n-s}(\mu_s^{(30)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} = \phi_{12,18}(\mu_{12}^{(30)})$, which is $< \omega_{30}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{29}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. \quad (9.18)$$

From (2.1), (2.2) and Claims (i), (ii) and (iii) we find that, $\max\{B_{17}, B_{18}, \dots, B_{29}\} < B_{17} \leq \frac{B_{29}}{\varepsilon^3} < 2.3985$. So using (9.18) we get that each of B_2, \dots, B_{13} is $< \max\{B_{14}, B_{15}, B_{16}, \frac{B_{29}}{\varepsilon^3}\} < \max\{B_{14}, B_{15}, B_{16}, 2.3985\}$.

Claim(v) $B_{14}, B_{15}, B_{16} < 2.3985$

We find that for $s = 15, 14, 13$, $\phi_{s,n-s}(\lambda_s^{(30)}) < \omega_{30}$, but $\phi_{s,n-s}(\mu_s^{(30)}) > \omega_{30}$, so we apply Lemma 11 respectively with $\sigma_{15}^{(30)} = (2.438)^{15}$, $\sigma_{14}^{(30)} = (2.305)^{16}$

and $\sigma_{13}^{(30)} = (2.185)^{17}$. Here $\phi_{15,15}(\sigma_{15}^{(30)}) < \omega_{30}$, $\phi_{14,16}(\sigma_{14}^{(30)}) < \omega_{30}$ and $\phi_{13,17}(\sigma_{13}^{(30)}) < \omega_{30}$.

First consider Lemma 11 for $s = 15$ and with $\sigma_{15}^{(30)} = (2.438)^{15}$.

In Case(i), when $B_1 B_2 \cdots B_{15} < (2.438)^{15}$, then we have $B_{16} < \max\{B_{17}, B_{18}, \dots, B_{30}\}$, which is $< \frac{B_{29}}{\epsilon^3} < 2.3985$.

In Case(ii), when $B_1 B_2 \cdots B_{15} \geq (2.438)^{15}$, then we have $B_{16} < \frac{\mu_{16}^{(30)}}{\sigma_{15}^{(30)}} < \frac{(2.7533)^{14}}{(2.438)^{15}} < 2.26$.

So we have $B_{16} < \frac{B_{29}}{\epsilon^3} < 2.3985$.

Now consider Lemma 11 for $s = 14$ and with $\sigma_{14}^{(30)} = (2.305)^{16}$.

In Case(i), when $B_1 B_2 \cdots B_{14} < (2.305)^{16}$, then we have $B_{15} < \max\{B_{16}, B_{17}, \dots, B_{30}\}$, which is $< \frac{B_{29}}{\epsilon^3} < 2.3985$.

In Case(ii), when $B_1 B_2 \cdots B_{14} \geq (2.305)^{16}$, then we have $B_{15} < \frac{\mu_{15}^{(30)}}{\sigma_{14}^{(30)}} < \frac{(2.5516)^{15}}{(2.305)^{16}} < 1.993$.

So we have $B_{15} < \frac{B_{29}}{\epsilon^3} < 2.3985$.

Further consider Lemma 11 for $s = 13$ and with $\sigma_{13}^{(30)} = (2.185)^{17}$.

In Case(i), when $B_1 B_2 \cdots B_{13} < (2.185)^{17}$, then we have $B_{14} < \max\{B_{15}, B_{16}, \dots, B_{30}\}$, which is $< \frac{B_{29}}{\epsilon^3} < 2.3985$.

In Case(ii), when $B_1 B_2 \cdots B_{13} \geq (2.185)^{17}$, then we have $B_{14} < \frac{\mu_{14}^{(30)}}{\sigma_{13}^{(30)}} < \frac{(2.36995)^{16}}{(2.185)^{17}} < 1.68$.

So we have $B_{14} < \frac{B_{29}}{\epsilon^3} < 2.3985$.

Using Claims(iv) and (v) we get each of B_2, B_3, \dots, B_{17} is $< \frac{B_{29}}{\epsilon^3} < 2.3985$.

Final Contradiction

Now $2B_2 + 2B_4 + \dots + 2B_{30} < 2(8 \times \frac{B_{29}}{\epsilon^3}) + 2(\frac{3/2}{\epsilon^2} + \frac{1}{\epsilon^2} + \frac{3/2}{\epsilon} + \frac{1}{\epsilon} + \frac{3}{2} + 1)B_{28} + 2B_{30} < 49.86$ for $B_{30} < 0.231$, $B_{29} < 0.247$ and $B_{28} < 0.285$, giving thereby a contradiction to the weak inequality $(2, 2, \dots, 2, 2)_w$. \square

9.16 $n = 31$

Here $\omega_{31} = 53.04$, $B_1 \leq \gamma_{31} < 4.8483483$. Using (2.5), we have $l_{31} = 0.0991 < B_{31} < 4.4974263 = m_{31}$.

Claim(i) $B_{31} < 0.173$

Suppose $B_{31} \geq 0.173$. The inequality $(30^*, 1)$ gives $49.86(B_{31})^{\frac{-1}{30}} + B_{31} > 53.04$. But this is not true for $0.173 \leq B_{31} \leq 4.4974263$. So we must have $B_{31} < 0.173$.

Claim(ii) $B_i < \max\{B_{15}, B_{16}, 2.5199\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned} 1 &= \lambda_1^{(31)} < B_1 < \mu_1^{(31)} = 4.8483483, \\ \frac{3}{4} &= \lambda_2^{(31)} < B_1 B_2 < \mu_2^{(31)} = (1.1120388)^{29}, \end{aligned}$$

$$\begin{aligned}
\frac{1}{2} &= \lambda_3^{(31)} < B_1 B_2 B_3 < \mu_3^{(31)} = (1.1744085)^{28}, \\
\frac{1}{4} &= \lambda_4^{(31)} < B_1 B_2 B_3 B_4 < \mu_4^{(31)} = (1.2415716)^{27}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(31)} < B_1 \cdots B_5 < \mu_5^{(31)} = (1.3140139)^{26}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(31)} < B_1 \cdots B_6 < \mu_6^{(31)} = (1.3922837)^{25}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(31)} < B_1 \cdots B_7 < \mu_7^{(31)} = (1.47700291)^{24}, \\
\frac{\varepsilon^4}{16} &= \lambda_8^{(31)} < B_1 \cdots B_8 < \mu_8^{(31)} = (1.5688842)^{23}, \\
\frac{\varepsilon^6}{16} &= \lambda_9^{(31)} < B_1 \cdots B_9 < \mu_9^{(31)} = (1.6687218)^{22}, \\
\frac{\varepsilon^{45}}{4^5 B_{31}^{21}} &= \lambda_{10}^{(31)} < B_1 \cdots B_{10} < \mu_{10}^{(31)} = (1.7774458)^{21}, \\
\frac{\varepsilon^{40}}{4^5 B_{31}^{20}} &= \lambda_{11}^{(31)} < B_1 \cdots B_{11} < \mu_{11}^{(31)} = (1.8961171)^{20}, \\
\frac{\varepsilon^{36}}{2 \times 4^4 B_{31}^{19}} &= \lambda_{12}^{(31)} < B_1 \cdots B_{12} < \mu_{12}^{(31)} = (2.0259616)^{19}, \\
\frac{3\varepsilon^{32}}{4^5 B_{31}^{18}} &= \lambda_{13}^{(31)} < B_1 \cdots B_{13} < \mu_{13}^{(31)} = (2.1684016)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{31}^{17}} &= \lambda_{14}^{(31)} < B_1 \cdots B_{14} < \mu_{14}^{(31)} = (2.3250968)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{31}^{16}} &= \lambda_{15}^{(31)} < B_1 \cdots B_{15} < \mu_{15}^{(31)} = (2.497994)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{31}^{15}} &= \lambda_{16}^{(31)} < B_1 \cdots B_{16} < \mu_{16}^{(31)} = (2.6893851)^{15}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(31)}), \phi_{s,n-s}(\mu_s^{(31)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} = \phi_{13,18}(\mu_1^{(31)})$, which is $< \omega_{31}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{31}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14. \quad (9.19)$$

From (2.1), (2.2) and Claim (i), we find that $\max\{B_{17}, B_{18}, \dots, B_{31}\} < B_{17} \leq \frac{3}{2} \frac{B_{31}}{\varepsilon^3} < 2.5199$. So using (9.19) we get that each of B_2, \dots, B_{14} is $< \max\{B_{15}, B_{16}, \frac{3}{2} \frac{B_{31}}{\varepsilon^3}\} < \max\{B_{15}, B_{16}, 2.5199\}$.

Claim(iii) $B_{15}, B_{16} < 2.5199$

We find that for $s = 15, 14$, $\phi_{s,n-s}(\lambda_s^{(31)}) < \omega_{31}$, but $\phi_{s,n-s}(\mu_s^{(31)}) > \omega_{31}$, so we apply Lemma 11 respectively with $\sigma_{15}^{(31)} = (2.42)^{16}$ and $\sigma_{14}^{(31)} = (2.299)^{17}$. Here $\phi_{15,16}(\sigma_{15}^{(31)}) < \omega_{31}$ and $\phi_{14,17}(\sigma_{14}^{(31)}) < \omega_{31}$.

First consider Lemma 11 for $s = 15$ and with $\sigma_{15}^{(31)} = (2.42)^{16}$.

In Case(i), when $B_1 B_2 \cdots B_{15} < (2.42)^{16}$, then we have $B_{16} < \max\{B_{17}, B_{18}, \dots, B_{31}\}$, which is $< \frac{3}{2} \frac{B_{31}}{\varepsilon^3} < 2.5199$.

In Case(ii), when $B_1 B_2 \cdots B_{15} \geq (2.42)^{16}$, then we have $B_{16} < \frac{\mu_{16}^{(31)}}{\sigma_{15}^{(31)}} < \frac{(2.6894)^{15}}{(2.42)^{16}} < 2.0128$.

So we have $B_{16} \leq \frac{3}{2} \frac{B_{31}}{\varepsilon^3} < 2.5199$.

Now consider Lemma 11 for $s = 14$ and with $\sigma_{14}^{(31)} = (2.299)^{17}$.

In Case(i), when $B_1 B_2 \cdots B_{14} < (2.299)^{17}$, then we have $B_{15} < \max\{B_{16}, B_{17}, \dots, B_{31}\}$, which is $\leq \frac{3}{2} \frac{B_{31}}{\varepsilon^3} < 2.5199$.

In Case(ii), when $B_1 B_2 \cdots B_{14} \geq (2.299)^{17}$, then we have $B_{15} < \frac{\mu_{15}^{(31)}}{\sigma_{14}^{(31)}} < \frac{(2.498)^{16}}{(2.299)^{17}} < 1.642$.

So we have $B_{15} \leq \frac{3}{2} \frac{B_{31}}{\epsilon^3} < 2.5199$.

Using Claims(ii) and (iii) we get each of B_2, B_3, \dots, B_{17} is $\leq \frac{3}{2} \frac{B_{31}}{\epsilon^3} < 2.5199$.

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + \dots + 2B_{31} < 3(\frac{3}{2} \frac{B_{31}}{\epsilon^3}) + 2(7 \times \frac{3/2}{\epsilon^3} + \frac{1}{\epsilon^3} + \frac{3/2}{\epsilon^2} + \frac{1}{\epsilon^2} + \frac{3/2}{\epsilon} + \frac{1}{\epsilon} + \frac{3}{2} + 1)B_{31} < 53.04$ for $B_{31} < 0.173$, giving thereby a contradiction to the weak inequality $(2, 1, 2, \dots, 2, 2)_w$. \square

9.17 $n = 32$

Here $\omega_{32} = 56.06$, $B_1 \leq \gamma_{32} < 4.9680344$. Using (2.5) we have $l_{32} = 0.0942 < B_{32} < 4.6149714 = m_{32}$.

Claim(i) $B_{32} < 0.203$

Suppose $B_{32} \geq 0.203$. The inequality $(31^*, 1)$ gives $53.04(B_{32})^{\frac{-1}{31}} + B_{32} > 56.06$. But this is not true for $0.203 \leq B_{32} \leq 4.6149714$. So we must have $B_{32} < 0.203$.

Claim(ii) $B_{30} < 0.261$

Suppose $B_{30} \geq 0.261$. The inequality $(29^*, 3)$ gives $47.18(B_{30}B_{31}B_{32})^{\frac{-1}{29}} + 4B_{30} - \frac{B_{30}^3}{B_{31}B_{32}} > 56.06$. But this is not true for $0.261 \leq B_{30} \leq \frac{3}{2}B_{32} < 0.305$ and $\frac{1}{2}B_{30}^2 < B_{31}B_{32} < \frac{4}{3}(0.203)^2$. So we must have $B_{30} < 0.261$.

Claim(iii) $B_i < \max\{B_{15}, B_{16}, B_{17}, 2.5344\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(32)} < B_1 < \mu_1^{(32)} = 4.9680344, \\
\frac{3}{4} &= \lambda_2^{(32)} < B_1B_2 < \mu_2^{(32)} = (1.1099695)^{30}, \\
\frac{1}{2} &= \lambda_3^{(32)} < B_1B_2B_3 < \mu_3^{(32)} = (1.1710561)^{29}, \\
\frac{1}{4} &= \lambda_4^{(32)} < B_1B_2B_3B_4 < \mu_4^{(32)} = (1.2367358)^{28}, \\
\frac{\epsilon}{4} &= \lambda_5^{(32)} < B_1 \dots B_5 < \mu_5^{(32)} = (1.3074634)^{27}, \\
\frac{3\epsilon^2}{16} &= \lambda_6^{(32)} < B_1 \dots B_6 < \mu_6^{(32)} = (1.3837502)^{26}, \\
\frac{\epsilon^3}{8} &= \lambda_7^{(32)} < B_1 \dots B_7 < \mu_7^{(32)} = (1.4661739)^{25}, \\
\frac{\epsilon^4}{16} &= \lambda_8^{(32)} < B_1 \dots B_8 < \mu_8^{(32)} = (1.5553893)^{24}, \\
\frac{\epsilon^6}{16} &= \lambda_9^{(32)} < B_1 \dots B_9 < \mu_9^{(24)} = (1.6521469)^{23}, \\
\frac{3\epsilon^{50}}{4^6 B_{32}^{22}} &= \lambda_{10}^{(32)} < B_1 \dots B_{10} < \mu_{10}^{(32)} = (1.7572829)^{22}, \\
\frac{\epsilon^{45}}{4^5 B_{32}^{21}} &= \lambda_{11}^{(32)} < B_1 \dots B_{11} < \mu_{11}^{(32)} = (1.8717771)^{21}, \\
\frac{\epsilon^{40}}{4^5 B_{32}^{20}} &= \lambda_{12}^{(32)} < B_1 \dots B_{12} < \mu_{12}^{(32)} = (1.9967464)^{20}, \\
\frac{\epsilon^{36}}{2 \times 4^4 B_{32}^{19}} &= \lambda_{13}^{(32)} < B_1 \dots B_{13} < \mu_{13}^{(32)} = (2.1334819)^{19}, \\
\frac{3\epsilon^{32}}{4^5 B_{32}^{18}} &= \lambda_{14}^{(32)} < B_1 \dots B_{14} < \mu_{14}^{(32)} = (2.2834814)^{18}, \\
\frac{\epsilon^{28}}{4^4 B_{32}^{17}} &= \lambda_{15}^{(32)} < B_1 \dots B_{15} < \mu_{15}^{(32)} = (2.4484926)^{17}, \\
\frac{\epsilon^{24}}{4^4 B_{32}^{16}} &= \lambda_{16}^{(32)} < B_1 \dots B_{16} < \mu_{16}^{(32)} = (2.6305651)^{16},
\end{aligned}$$

$$\frac{\varepsilon^{21}}{2 \times 4^3 B_{32}^{15}} = \lambda_{17}^{(32)} < B_1 \cdots B_{17} < \mu_{17}^{(32)} = (2.8321145)^{15}.$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(32)}), \phi_{s,n-s}(\mu_s^{(32)})\}$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} = \phi_{1,31}(\mu_1^{(32)})$, which is $< \omega_{32}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{32}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14. \quad (9.20)$$

From (2.1), (2.2) and Claims (i), (ii), we find that $\max\{B_{18}, B_{19}, \dots, B_{32}\} < B_{18} \leq \frac{B_{30}}{\varepsilon^3} < 2.5344$. So using (9.20) we get that each of B_2, \dots, B_{14} is $< \max\{B_{15}, B_{16}, B_{17}, \frac{B_{30}}{\varepsilon^3}\} < \max\{B_{15}, B_{16}, B_{17}, 2.5344\}$.

Claim(iv) $B_{15}, B_{16}, B_{17} < 2.5344$

We find that for $s = 16, 15, 14$, $\phi_{s,n-s}(\lambda_s^{(32)}) < \omega_{32}$, but $\phi_{s,n-s}(\mu_s^{(32)}) > \omega_{32}$, so we apply Lemma 11 respectively with $\sigma_{16}^{(32)} = (2.506)^{16}$, $\sigma_{15}^{(32)} = (2.4)^{17}$ and $\sigma_{14}^{(32)} = (2.27)^{18}$. Here $\phi_{16,16}(\sigma_{16}^{(32)}) < \omega_{32}$, $\phi_{15,17}(\sigma_{15}^{(32)}) < \omega_{32}$ and $\phi_{14,18}(\sigma_{14}^{(32)}) < \omega_{32}$.

First consider Lemma 11 for $s = 16$ and with $\sigma_{16}^{(32)} = (2.506)^{16}$.

In Case(i), when $B_1 B_2 \cdots B_{16} < (2.506)^{16}$, then we have $B_{17} < \max\{B_{18}, B_{19}, \dots, B_{32}\}$, which is $\leq \frac{B_{30}}{\varepsilon^3} < 2.5344$.

In Case(ii), when $B_1 B_2 \cdots B_{16} \geq (2.506)^{16}$, then we have $B_{17} < \frac{\mu_{17}^{(32)}}{\sigma_{16}^{(32)}} < \frac{(2.833)^{15}}{(2.506)^{16}} < 2.512$.

So we have $B_{17} < \frac{B_{30}}{\varepsilon^3} < 2.5344$.

Now consider Lemma 11 for $s = 15$ and with $\sigma_{15}^{(32)} = (2.4)^{17}$.

In Case(i), when $B_1 B_2 \cdots B_{15} < (2.4)^{17}$, then we have $B_{16} < \max\{B_{17}, B_{18}, \dots, B_{32}\}$, which is $< \frac{B_{30}}{\varepsilon^3} < 2.5344$.

In Case(ii), when $B_1 B_2 \cdots B_{15} \geq (2.4)^{17}$, then we have $B_{16} < \frac{\mu_{16}^{(32)}}{\sigma_{15}^{(32)}} < \frac{(2.6306)^{16}}{(2.4)^{17}} < 1.809$.

So we have $B_{16} < \frac{B_{30}}{\varepsilon^3} < 2.5344$.

Next consider Lemma 11 for $s = 14$ and with $\sigma_{14}^{(32)} = (2.27)^{18}$.

In Case(i), when $B_1 B_2 \cdots B_{14} < (2.27)^{18}$, then we have $B_{15} < \max\{B_{16}, B_{17}, \dots, B_{32}\}$, which is $< \frac{B_{30}}{\varepsilon^3} < 2.5344$.

In Case(ii), when $B_1 B_2 \cdots B_{14} \geq (2.27)^{18}$, then we have $B_{15} < \frac{\mu_{15}^{(32)}}{\sigma_{14}^{(32)}} < \frac{(2.4485)^{17}}{(2.27)^{18}} < 1.596$.

So we have $B_{15} < \frac{B_{30}}{\varepsilon^3} < 2.5344$.

Using Claims(iii) and (iv) we get each of B_2, B_3, \dots, B_{18} is $< \frac{B_{30}}{\varepsilon^3} < 2.5344$.

Final Contradiction

Now $2B_2 + 2B_4 + 2B_6 + \cdots + 2B_{32} < 2(9 \times \frac{1}{\varepsilon^3} + \frac{3/2}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{30} + 2B_{32} < 56.06$ for $B_{32} < 0.203$ and $B_{30} < 0.261$, giving thereby a contradiction to the weak inequality $(2, 2, 2, \dots, 2, 2)_w$. \square

9.18 $n = 33$

Here $\omega_{33} = 59.58$, $B_1 \leq \gamma_{33} < 5.0876409$. Using (2.5), we have $l_{33} = 0.0896 < B_{33} < 4.7324725 = m_{33}$.

Claim(i) $B_{33} < 0.155$

Suppose $B_{33} \geq 0.155$. The inequality $(32^*, 1)$ gives $56.06(B_{33})^{\frac{-1}{32}} + B_{33} > 59.58$. But this is not true for $0.155 \leq B_{33} < 4.7324725$.

Claim(ii) $B_i < \max\{B_{16}, B_{17}, B_{18}, 2.2577\}$, for $i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$

Using (2.4) and Lemmas 8, 9 we have

$$\begin{aligned}
1 &= \lambda_1^{(33)} < B_1 < \mu_1^{(33)} = 5.0876409, \\
\frac{3}{4} &= \lambda_2^{(33)} < B_1 B_2 < \mu_2^{(33)} = (1.1079939)^{31}, \\
\frac{1}{2} &= \lambda_3^{(33)} < B_1 B_2 B_3 < \mu_3^{(33)} = (1.1678596)^{30}, \\
\frac{1}{4} &= \lambda_4^{(33)} < B_1 B_2 B_3 B_4 < \mu_4^{(33)} = (1.2321321)^{29}, \\
\frac{\varepsilon}{4} &= \lambda_5^{(33)} < B_1 B_2 B_3 B_4 B_5 < \mu_5^{(33)} = (1.3012373)^{28}, \\
\frac{3\varepsilon^2}{16} &= \lambda_6^{(33)} < B_1 B_2 B_3 B_4 B_5 B_6 < \mu_6^{(33)} = (1.3756537)^{27}, \\
\frac{\varepsilon^3}{8} &= \lambda_7^{(33)} < B_1 \cdots B_7 < \mu_7^{(33)} = (1.4559193)^{26}, \\
\frac{\varepsilon^4}{16} &= \lambda_8^{(33)} < B_1 \cdots B_8 < \mu_8^{(33)} = (1.5426418)^{25}, \\
\frac{\varepsilon^6}{16} &= \lambda_9^{(33)} < B_1 \cdots B_9 < \mu_9^{(24)} = (1.6365102)^{24}, \\
\frac{\varepsilon^{55}}{2 \times 4^5 B_{33}^{23}} &= \lambda_{10}^{(33)} < B_1 \cdots B_{10} < \mu_{10}^{(33)} = (1.7383141)^{23}, \\
\frac{3\varepsilon^{50}}{4^6 B_{33}^{22}} &= \lambda_{11}^{(33)} < B_1 \cdots B_{11} < \mu_{11}^{(33)} = (1.8489335)^{22}, \\
\frac{\varepsilon^{45}}{4^5 B_{33}^{21}} &= \lambda_{12}^{(33)} < B_1 \cdots B_{12} < \mu_{12}^{(33)} = (1.9693991)^{21}, \\
\frac{\varepsilon^{40}}{4^5 B_{33}^{20}} &= \lambda_{13}^{(33)} < B_1 \cdots B_{13} < \mu_{13}^{(33)} = (2.1008861)^{20}, \\
\frac{\varepsilon^{36}}{2 \times 4^4 B_{33}^{19}} &= \lambda_{14}^{(33)} < B_1 \cdots B_{14} < \mu_{14}^{(33)} = (2.24475296)^{19}, \\
\frac{3\varepsilon^{32}}{4^5 B_{33}^{18}} &= \lambda_{15}^{(33)} < B_1 \cdots B_{15} < \mu_{15}^{(33)} = (2.4025756)^{18}, \\
\frac{\varepsilon^{28}}{4^4 B_{33}^{17}} &= \lambda_{16}^{(33)} < B_1 \cdots B_{16} < \mu_{16}^{(33)} = (2.5761882)^{17}, \\
\frac{\varepsilon^{24}}{4^4 B_{33}^{16}} &= \lambda_{17}^{(33)} < B_1 \cdots B_{17} < \mu_{17}^{(33)} = (2.7677614)^{16}, \\
\frac{\varepsilon^{21}}{2 \times 4^3 B_{33}^{15}} &= \lambda_{18}^{(33)} < B_1 \cdots B_{18} < \mu_{18}^{(33)} = (2.9798226)^{15}.
\end{aligned}$$

We find that $\max\{\phi_{s,n-s}(\lambda_s^{(33)}), \phi_{s,n-s}(\mu_s^{(33)})$, for $s = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\} = \phi_{14,19}(\mu_{14}^{(33)})$, which is $< \omega_{33}$. Therefore using Lemma 10 we have

$$B_i < \max\{B_{i+1}, B_{i+2}, \dots, B_{33}\}, \text{ for } i = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. \quad (9.21)$$

From (2.1), (2.2) and Claim(i) we find that, $\max\{B_{19}, B_{20}, \dots, B_{33}\} < B_{19} \leq \frac{3}{2} \frac{B_{33}}{\varepsilon^3} < 2.2577$. So using (9.21) we get that each of B_2, \dots, B_{15} is $< \max\{B_{16}, B_{17}, B_{18}, \frac{3}{2} \frac{B_{33}}{\varepsilon^3}\} < \max\{B_{16}, B_{17}, B_{18}, 2.2577\}$.

Claim(iii) $B_{16}, B_{17}, B_{18} < 2.7$

We find that for $s = 17, 16, 15$, $\phi_{s,n-s}(\lambda_s^{(33)}) < \omega_{33}$, but $\phi_{s,n-s}(\mu_s^{(33)}) > \omega_{33}$, so we apply Lemma 11 respectively with $\sigma_{17}^{(33)} = (2.616)^{16}$, $\sigma_{16}^{(33)} = (2.5)^{17}$

and $\sigma_{15}^{(33)} = (2.39)^{18}$. Here $\phi_{17,16}(\sigma_{17}^{(33)}) < \omega_{33}$, $\phi_{16,17}(\sigma_{16}^{(33)}) < \omega_{33}$ and $\phi_{15,18}(\sigma_{15}^{(33)}) < \omega_{33}$.

First consider Lemma 11 for $s = 17$ and with $\sigma_{17}^{(33)} = (2.616)^{16}$.

In Case(i), when $B_1 B_2 \cdots B_{17} < (2.616)^{16}$, then we have $B_{18} < \max\{B_{19}, B_{20}, \dots, B_{33}\}$, which is $< \frac{3}{2} \frac{B_{33}}{\varepsilon^3} < 2.2577$.

In Case(ii), when $B_1 B_2 \cdots B_{17} \geq (2.616)^{16}$, then we have $B_{18} < \frac{\mu_{18}^{(33)}}{\sigma_{17}^{(33)}} < \frac{(2.9797)^{15}}{(2.616)^{16}} < 2.7$.

So we have $B_{18} < 2.7$.

Now consider Lemma 11 for $s = 16$ and with $\sigma_{16}^{(33)} = (2.5)^{17}$.

In Case(i), when $B_1 B_2 \cdots B_{16} < (2.5)^{17}$, then we have $B_{17} < \max\{B_{18}, B_{19}, \dots, B_{33}\}$, which is < 2.7 .

In Case(ii), when $B_1 B_2 \cdots B_{16} \geq (2.5)^{17}$, then we have $B_{17} < \frac{\mu_{17}^{(33)}}{\sigma_{16}^{(33)}} < \frac{(2.7678)^{16}}{(2.5)^{17}} < 2.038$.

So we have $B_{17} < 2.7$.

Next consider Lemma 11 for $s = 15$ and with $\sigma_{15}^{(33)} = (2.39)^{18}$.

In Case(i), when $B_1 B_2 \cdots B_{15} < (2.39)^{18}$, then we have $B_{16} < \max\{B_{17}, B_{18}, \dots, B_{33}\}$, which is < 2.7 .

In Case(ii), when $B_1 B_2 \cdots B_{15} \geq (2.39)^{18}$, then we have $B_{16} < \frac{\mu_{16}^{(33)}}{\sigma_{15}^{(33)}} < \frac{(2.5762)^{17}}{(2.39)^{18}} < 1.498$.

So we have $B_{16} < 2.7$.

Using Claims(ii) and (iii) we get each of B_2, B_3, \dots, B_{18} is < 2.7 .

Final Contradiction

Now $2B_2 + B_3 + 2B_5 + 2B_7 + \cdots + 2B_{33} < 3 \times 2.7 + 2(7 \times 2.7) + 2(\frac{3/2}{\varepsilon^3} + \frac{1}{\varepsilon^3} + \frac{3/2}{\varepsilon^2} + \frac{1}{\varepsilon^2} + \frac{3/2}{\varepsilon} + \frac{1}{\varepsilon} + \frac{3}{2} + 1)B_{33} < 59.58$ for $B_{33} < 0.155$, giving thereby a contradiction to the weak inequality $(2, 1, 2, \dots, 2, 2)_w$. \square

References

- [1] R. P. Bambah, V. C. Dumir and R. J. Hans-Gill, *Non-homogeneous problems: Conjectures of Minkowski and Watson*, Number Theory, Trends in Mathematics, Birkhauser Verlag, Basel, (2000) 15-41.
- [2] B. J. Birch and H. P. F. Swinnerton-Dyer, *On the inhomogeneous minimum of the product of n linear forms*, Mathematika 3 (1956), 25-39.
- [3] H. F. Blichfeldt, *The minimum values of positive quadratic forms in six, seven and eight variables*, Math. Z. 39 (1934), 1-15.
- [4] N. Čebotarev, *Beweis des Minkowski'schen Satzes über lineare inhomogene Formen*, Vierteljschr. Naturforsch. Ges. Zurich, 85 Beiblatt, (1940), 27-30.

- [5] H. Cohn and N. Elkies, *New upper bounds on sphere packings*, I. Ann. of Math., 157(2) (2003), 689-714.
- [6] H. Cohn and A. Kumar, *The densest lattice in twenty-four dimensions*, Electron. Res. Announc. Amer. Math. Soc. 10 (2004), 58-67.
- [7] J. H. Conway and N. J. A. Sloane, *Sphere packings, Lattices and groups*, Springer-Verlag, Second edition, New York, 1993.
- [8] P. Gruber, *Convex and discrete geometry*, Springer Grundlehren Series (vol.336) 2007.
- [9] P. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, Second Edition, North Holland, 37 (1987).
- [10] R. J. Hans-Gill, Madhu Raka, Ranjeet Sehmi and Sucheta, *A unified simple proof of Woods' conjecture for $n \leq 6$* , J. Number Theory, 129 (2009) 1000-1010.
- [11] R. J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, *On conjectures of Minkowski and Woods for $n = 7$* , J. Number Theory, 129 (2009), 1011-1033.
- [12] R. J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, *Estimates On Conjectures of Minkowski and Woods*, Indian Jl. Pure Appl. Math., 41(4) (2010), 595-606.
- [13] R.J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, *On Conjectures of Minkowski and Woods for $n = 8$* , Acta Arithmetica, 147(4) (2011), 337-385.
- [14] R. J. Hans-Gill, Madhu Raka and Ranjeet Sehmi, *Estimates On Conjectures of Minkowski and Woods II*, Indian Jl. Pure Appl. Math., 42(5) (2011), 307-333.
- [15] I.V. Il'in, *A remark on an estimate in the inhomogeneous Minkowski conjecture for small dimensions*, (Russian) 90, Petrozavodsk. Gos. Univ., Petrozavodsk, (1986), 24-30.
- [16] I.V. Il'in, *Čhebotarev estimates in the inhomogeneous Minkowski conjecture for small dimensions*, Algebraic systems, Ivanov. Gos. Univ., Ivanovo, (1991), 115-125.
- [17] Leetika Kathuria and Madhu Raka, *On Conjectures of Minkowski and Woods for $n = 9$* , arXiv:1410.5743v1 [math.NT], 21 Oct, 2014.
- [18] Leetika Kathuria and Madhu Raka, *Generalization of a result of Birch and Swinnerton-Dyer*, Submitted for publication

- [19] A. Korkine, G. Zolotareff, *Sur les formes quadratiques*, Math. Ann. 6 (1873), 366-389; *Sur les formes quadratiques positives*, Math. Ann. 11 (1877), 242-292.
- [20] C. T. McMullen, *Minkowski's conjecture, well rounded lattices and topological dimension*, J. Amer. Math. Soc. 18 (2005), 711-734.
- [21] L. J. Mordell, *Tschebotareff's Theorem on the product of Non-homogeneous Linear Forms (II)*, J. London Math Soc. 35 (1960), 91-97.
- [22] R.A. Pendavingh and S.H.M. Van Zwam, *New Korkine-Zolotarev inequalities*, SIAM J. Optim. 18 (2007), no. 1, 364-378.
- [23] A. C. Woods, *The densest double lattice packing of four spheres*, Mathematika 12 (1965) 138-142.
- [24] A. C. Woods, *Lattice coverings of five space by spheres*, Mathematika 12 (1965) 143-150.
- [25] A. C. Woods, *Covering six space with spheres*, J. Number Theory 4 (1972) 157-180.